Fuzzy Linguistic Signatures

Nour Ammar and László T. Kóczy

***Abstract*—Fuzzy Linguistic Signatures (FLS) extend the concept of Fuzzy Signatures (FSigs) by introducing linguistic variables as qualitative descriptors within a hierarchical fuzzy structure. Although fuzzy signatures have been successfully applied in various domains, their reliance on numerical membership degrees limits their ability to model subjective or linguistically defined information. This paper establishes a formal mathematical frame-work for FLS by defining a family of fuzzy linguistic signatures equipped with suitable linguistic aggregation operators and a partial ordering relation among linguistic values. Furthermore, meet-and-join operators are introduced to demonstrate that FLS satisfies the properties of a lattice as an algebraic structure. Consequently, fuzzy linguistic signatures provide an expressive representational framework capable of handling qualitative, human-like reasoning.**

***Index Terms*—Fuzzy Sets; Fuzzy Signature; Fuzzy Linguistic Signature**

I. INTRODUCTION

Fuzzy Signatures are multi-component fuzzy descriptors, extensions of the original concept of fuzzy set [1] and of Vector Valued Fuzzy Sets [5], with multi-level nested structure, where sub-components may be arranged in sub-signatures, according to closer interdependence or other ways of connectedness. Fuzzy sets are defined as follows:

$$A_f : \langle X, \mu_A : X \to [0,1]\rangle. \quad (1)$$

A vector valued fuzzy set (VFF set) is an extension of the above:

$$A_f^v : \langle X,\ \boldsymbol{\mu}^v : X \to [0,1]^n\rangle. \quad (2)$$

Here, the membership function maps each element of X into an n-component vector, where each component is an element of [0, 1]. This type of extended fuzzy set can be used when there is a multitude of properties for the same elements, and the i^{th} component of the membership vector expresses the degree of belonging of the VFF set in the sense of the i^{th} property.

Fuzzy Signatures (FSigs) are further extensions, namely, where the components of the fuzzy membership degree vector may be arbitrary multi-dimensional vectors themselves, thereby constructing a multilevel nested hierarchy of membership degrees. FSigs may be conveniently represented either by the mentioned nested vectorial structure or by a rooted tree graph, where each nested vector corresponds to a subtree, and the actual membership degrees are assigned to the leaves. Equation (3) defines fuzzy signatures recursively: each component μ_i may be either a scalar membership degree in $[0,1]$ or a vector-valued membership function of the same form, thereby inducing a finite rooted tree structure.:

$$A_f^{\text{sig}} : \langle X \to \boldsymbol{\mu}^{\text{sig}}\rangle, \quad \boldsymbol{\mu}^{\text{sig}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{bmatrix}. \quad (3)$$

Where:

$$\mu_i = \left\{ \begin{array}{c} \mu_i \\ \text{or} \\ \begin{bmatrix} \mu_{i1} \\ \mu_{i2} \\ \mu_{i3} \\ \vdots \\ \mu_{im} \end{bmatrix} \end{array} \right\} \quad \text{and so on, recursively.}$$

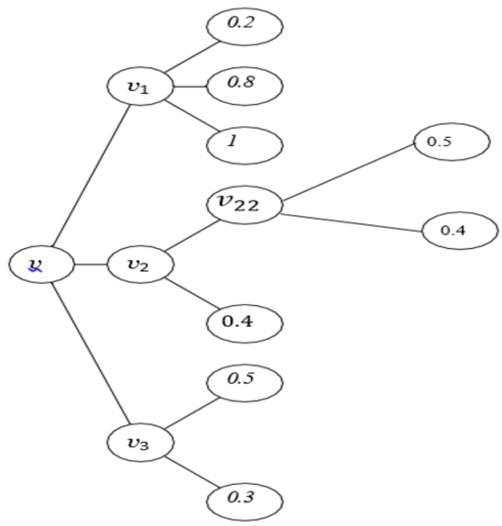

Fig. 1: Rooted tree for a FSigs

At the leaves, there may be membership functions of the respective universe rather than a single membership degree. For an exact mathematical definition and the algebraic structure of fuzzy signatures, a special case of L-fuzzy objects/sets, see [1]. Some aspects of the potential applications may be found in [2], and the specific problem of modeling the traffic situation in road networks was proposed in [3]. Linguistic variables have been applied by Wong [2] to describe uncertain values where even the exact fuzzy membership functions are hard to

N. Ammar is with the Department of Telecommunications and Artificial Intelligence, Budapest University of Technology and Economics, Budapest, Hungary (e-mail: nour.ammar@edu.bme.hu).

L. T. Koczy is with the Department of Information Technology, Szechenyi Istvan University, Gyor, Hungary (e-mail: koczy@tmit.bme.hu)

DOI: 10.36244/ICJ.2026.1.5

determine. The idea was first proposed by Zadeh [4] but was later discussed by numerous publications [16].

However, in some cases, assigning membership values and membership functions to the leaves is not feasible, thus they require an extension of this structure. For instance, in [7] the study aimed to analyze employee behavior, including both organizational citizenship behavior (OCB) and counter-productive work behavior (CWB). The study includes many subjective components that cannot be easily represented in a traditional, quantitative manner (e.g. altruism, courtesy and complaints).

Thus, we assigned linguistic labels to these components rather than membership degrees, where linguistic terms can characterize the values of these variables, will be more suitable to tackle this problem. This is the motivation to introduce Fuzzy Linguistic Signatures (FLS).

II. The Concept of Linguistic Fuzzy Signatures

Fuzzy systems were originally introduced to model systems and situations with non-probabilistic uncertainty (vagueness). However, in some areas, even the determination of fuzzy membership degrees and membership functions are not possible as inputs and outputs of such a system. So, they are formulated in natural language terms and expressions. To address this issue, Zadeh first proposed the concept of fuzzy sets and fuzzy systems, later introduced an even more human-friendly modeling technique, namely linguistic variables [4].

According to Zadeh, a linguistic variable is a variable whose values are words or sentences in a natural language rather than numbers. It can be characterized by the quintuple-tuple $\langle X, T(X), U, G, M \rangle$, where X is the variable; $T(X)$ is the term set of X; U is the universe of discourse; G is a syntactic rule set that generates the terms in $T(X)$; and M is a semantic rule set that associates a meaning with each linguistic value l (i.e., $M(l)$ denotes a fuzzy subset of U). For example, the term set of the linguistic variable Age may be $T(\mathrm{Age}) = \{\text{very young}, \text{ young}, \text{ not young}, \text{ extremely old}\}$ [4]. Such linguistic-variable-based representations have been successfully employed in practical optimization and decision models, including fuzzy assignment models formulated using linguistic variables [17], [18].

Definition 1: Linguistic Variable

The above definition of linguistic variable may be simplified by omitting G from the model, as the syntax of the model will be restricted to a set of production rules, which may be defined and represented in the form of lookup tables (multidimensional tables) for associating output values to the input combinations. Thus, the model may be reduced to a quadruple $\langle X, T(X), U, M \rangle$.

Let us present a simple example. Let X be the linguistic variable "Weather" with the three-value term set $T(X) = \{\text{rainy}, \text{overcast}, \text{sunny}\}$. In this context, U may be the universe containing the percentage of the sky covered by clouds in Cartesian product with the amount of precipitation. This seems to be a reasonable formalization that helps map U to the unit interval and thus generate a set of fuzzy membership functions, formally by the mapping M. The meaning of a linguistic value l is characterized by a membership function, which associates its degree of compatibility with each $u \in U$ with the linguistic value l [14].

In the proposed structure, the structure of the rooted tree itself represents the hierarchical organization of fuzzy linguistic signatures (FLS) and is intentionally kept identical to that of fuzzy signatures (FSigs). This design ensures structural compatibility and allows direct comparison with the existing fuzzy signature model. The novelty of FLS lies in the semantic and algebraic nature of the linguistic values assigned to the leaves and propagated through the hierarchy. Another essential component of the FLS structure is the set of linguistic aggregation operators assigned to the internal nodes.

The paper proposes a definition of fuzzy linguistic aggregation operators that differs from earlier approaches in the literature. In FLS, numerical membership degrees are replaced by linguistic labels, which constitute a partially ordered set and cannot, in general, be embedded into a numerical scale without loss of meaning. To ensure this, aggregation at internal nodes must be defined in a manner that is entirely linguistic and does not rely on any numerical interpretation or mapping.

In the next section, a representation and reasoning framework is presented that is fundamentally different from FSigs, opening the door to computing with words [8] ,[15] within a hierarchical structure. The key question that follows is what kind of aggregation operators are capable of combining such linguistic values into a single linguistic result while satisfying the exact mathematical requirements?

A. Aggregation Operators for Fuzzy Linguistic Signatures

As mentioned above, Fuzzy Linguistic Signatures (FLS) are represented by a tree structure where the leaves are assigned linguistic values or labels, and intermediate nodes represent the aggregation result for the leaves. In this paragraph, we will review the definition of fuzzy aggregation operators and partial order among linguistic values and then will address the concept of linguistic aggregation as it appears in the literature and attempt to formulate the potential approaches for linguistic aggregations within FLS.

Definition 2: Aggregation Operators

Let $(P, \leq, \perp, \top)$ be a bounded partially ordered set (poset), where $\perp$ and $\top$ denote the lower bound and upper bound elements of P, respectively, such that

$$\perp \leq p \leq \top \quad \text{for all } p \in P.$$

An n-argument aggregation operator $a : P^n \to P$ is an order-preserving operator satisfying the following conditions:

1) $a(\top, \ldots, \top) = \top$.
2) $a(\bot, \ldots, \bot) = \bot$.
3) If $x_i \leq y_i$ for all i, then

$$a(x_1, \ldots, x_n) \leq a(y_1, \ldots, y_n).$$

The first and second conditions are called boundary conditions, and the third resembles the monotonicity property of the operator.

Definition 3: Partial Order of Linguistic Values

Let us assume we have a set of linguistic labels $L = \{L_1, L_2, \ldots, L_n\}$, where each label is represented by a vector:

$$\begin{aligned} L_1 &= \{L_{11}, L_{12}, \ldots, L_{1i}\} \\ L_2 &= \{L_{21}, L_{22}, \ldots, L_{2j}\} \\ L_3 &= \{L_{31}, L_{32}, \ldots, L_{3k}\} \\ &\vdots \\ L_n &= \{L_{n1}, L_{n2}, \ldots, L_{nm}\} \end{aligned}$$

The set L is partially ordered if there exists a relation $\leq$ for all pairs within L, and this relation satisfies the following properties:

1) $(i_1, j_1, k_1, \ldots, n_1) \leq (i_2, j_2, k_2, \ldots, n_2)$ $\Leftrightarrow i_1 \leq i_2,\ j_1 \leq j_2,$ and $k_1 \leq k_2, \ldots, n_1 \leq n_2$.

2) They have an upper bound $(i_1, j_1, k_1, \ldots, n_1)\,, \quad (i_2, j_2, k_2, \ldots, n_2) \leq (\max\{i_1, i_2\}, \max\{j_1, j_2\}, \max\{k_1, k_2\}, \ldots, \max\{n_1, n_2\})$

3) and lower bound: $(i_1, j_1, k_1, \ldots, n_1), (i_2, j_2, k_2, \ldots, n_2) \geq (\min\{i_1, i_2\}, \min\{j_1, j_2\}, \min\{k_1, k_2\}, \ldots, \min\{n_1, n_2\})$

   Let us discuss a simple example for illustrating the partial order where not all the labels are directly comparable. In this context, we use the supremum and infimum operators to combine linguistic labels. Throughout this section, we assume that the set of linguistic labels is finite and equipped with a bounded componentwise partial order. Under this assumption, every finite subset admits a well-defined infimum and supremum, given by the componentwise minimum and maximum, respectively. Consider two linguistic sets describing temperature and humidity where Tem ={"Cold", "Medium", "Warm"} and Humidity={"Dry", "Moderate", "Humid"}.

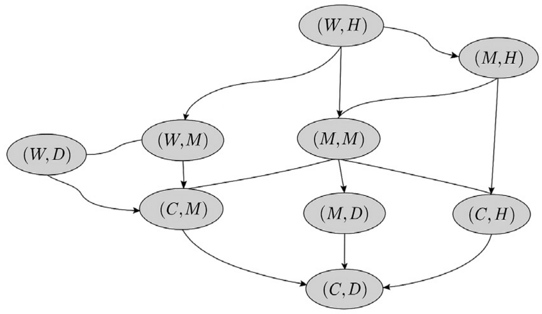

Fig. 2: Partial order of humidity and temperatures

Figure 2 illustrates the partial order defined on the Cartesian product of the linguistic variables temperature and humidity. The tuple (C, D), corresponding to {Cold, Dry}, represents the lower bound of the set, while the tuple (W, H), corresponding to {Warm, Humid}, represents the upper bound. Tuples such as (W, D), corresponding to {Warm, Dry}, and (M, M), corresponding to {Medium, Moderate}, are incomparable, although they share the same upper and lower bounds, because the partial order on the Cartesian product is defined component-wise and neither tuple dominates the other in all dimensions.

Linguistic aggregations have a very wide literature; an excellent overview is given in [9]. Numbers are assigned to the linguistic labels; the 2-tuple over weighted aggregation (TOWA), which is based on aggregation, utilizes the Extension Principle rather than relying solely on ordinal numbers [9]. Further extensions of the OWA by Xu [10], among others, follow this principle. All these operators share two common features: they are based on the ordinal numbers assigned to the linguistic labels, and they are related to the fuzzy OWA operators by Yager [11].

While the importance of OWA aggregation is undeniable, it should be noted that fuzzy aggregation includes several other important types. Fuzzy t-norms (intersections), t-conorms (unions), and various types of mean operations (arithmetic, geometric, harmonic, power means, etc.) also fall under this category, with the OWA being a special case of these means. Additionally, all kinds of hybrids may be defined, with the only condition being that they satisfy the original definition of fuzzy aggregations. Let us here recall the definition of fuzzy aggregation.

Definition 4: Fuzzy Aggregation Operators

Let $a(x_1, x_2, \ldots, x_n)$, where $x_i \in [0, 1]$, be a fuzzy function. Then a is an aggregation if and only if

$$a(0, 0, \ldots, 0) = 0, \quad a(1, 1, \ldots, 1) = 1,$$
$$a(x_1, x_2, \ldots, x_n) \geq a(y_1, y_2, \ldots, y_n) \quad \text{iff } x_i \geq y_i \text{ for all } i.$$

So, $a(x_1, x_2, \ldots, x_n)$ is a fuzzy aggregation operator over $x_1, x_2, \ldots, x_n$.

There is no argument against defining a concrete aggregation operations in a way that utilizes ordinal numbers (possibly tuples), but it is not a necessity, even in the case of finite and fixed sets of linguistic labels, and other ways, as e.g., by definition using rule-based system, defining the finite possible combinations, is an alternative. However, such ordinal number-based definitions cease to be applicable if the set of possible linguistic values is not yet fixed or the linguistic labels satisfy only partial ordering (no linear ordering). In natural languages, there are almost infinite possibilities to express fine differences among values, as e.g., linguistic hedges may modify the meaning of a basic label. If "expensive" is defined, "very expensive" may be considered a separate linguistic value, but "very-very expensive" has no meaning in this closed system. If there are fuzzy sets (in practice, convex and normal fuzzy sets, such as fuzzy numbers or fuzzy intervals) behind each label, representation, and calculations may become more complicated, but the set of possible labels may be kept "open" for possible additional linguistic values occurring during the operation of the modeled system, or when specifications and requirements are changed. A new, more "linguistic" definition could include such possibilities as well and also multi-component linguistic descriptors like "fast and expensive" or "slow and cheap". This way, much wider application fields for computing with words are opened [9], [15].

However, before discussing this new class of operations, it should be stated that it is reasonable to keep the signatures within a similar mathematical framework as FSigs of a family were defined in [1], as this would guarantee that signatures can be combined with each other by fuzzy (or similar) operations, that they can be compared, and that other manipulations necessary for further processing the given (linguistic fuzzy) data may be executed while preserving the exact mathematical justification for why and how to do this. As FSigs of the same family form a lattice, it would be advantageous to keep this essential property also for FLSs as well.

In order to determine the lattice structure of FLSs, the main point is to define the concept of linguistic aggregations in a way that preserves the properties forming the base of FSigs of a family being a lattice. In the next section, a simple extension for linguistic aggregations is given.

Definition 5: Linguistic Aggregation

Let

$$a_L(x_1, x_2, \ldots, x_n) \in L = \{L_\downarrow, L_1, L_2, \ldots, L_m, L_\uparrow\}$$

be a mapping

$$a_L : L^n \to L,$$

where $x_1, x_2, \ldots, x_n \in L$ are linguistic variables. Furthermore,

$$L_\downarrow \leq L_i \leq L_\uparrow, \quad \text{for all } i = 1, 2, \ldots, m,$$

and $\leq$ denotes a partial ordering on L.

L is the set of linguistic labels where $L_\downarrow$ is the lower bound and $L_\uparrow$ is the upper bound of the elements of L in the sense of $\leq$. Then a_L is a linguistic aggregation over L if and only if:

$$a_L(L_\downarrow, L_\downarrow, \ldots, L_\downarrow) = L_\downarrow,$$
$$a_L(L_\uparrow, L_\uparrow, \ldots, L_\uparrow) = L_\uparrow,$$
$$a_L(x_1, x_2, \ldots, x_n) \leq a_L(y_1, y_2, \ldots, y_n),$$
$$\text{for all } x_i \leq y_i \in L,$$

in the sense of the partial ordering.

Definition 6: Family of Linguistic Aggregation Operators

Let a family of linguistic aggregations over L, denoted by A_L, be defined as a set of all aggregation operators:

$$A_L = \{a_1, a_2, a_3, \ldots, a_n \mid a_i : L^n \to L\}.$$

Here, A_L is partially ordered by $\leq$, where for any two operators $a_1, a_2 \in A_L$:

$a_1 \leq a_2 \;\Leftrightarrow\; a_1(x_1, \ldots, x_n) \leq a_2(x_1, \ldots, x_n),$
$for \quad all(x_1, \ldots, x_n).$

With this ordering $(A_L, \leq)$, this pair is called the family of aggregators, and there exist a lower and upper bound (infimum and supremum) associated with this ordering. These are denoted as $\inf_L$ and $\sup_L$, respectively.

One of the simplest and non-trivial families is given by the set of aggregation operators:

$$A_L = \{a_{\inf}, a_x, a_y, a_{\sup}\},$$

where:

- $a_{\inf}$: returns the greatest lower bound (infimum) of the inputs (e.g., "low");
- $a_{\sup}$: returns the least upper bound (supremum) of the inputs, according to the partial order over L (e.g., "high");
- a_x and a_y: identity operators that simply return the linguistic values x and y.

To establish a suitable aggregation operation for linguistic variables within the fuzzy linguistic signature (FLS) framework, it is essential to define a partial ordering relation among the linguistic variables from the same family. This ordering provides a structured way to rank linguistic terms (such as "low," "medium," and "high") so that aggregation operators can meaningfully combine them.

Figure 3 shows an example of how these operators can be organized in a complete lattice, allowing linguistic terms to be ordered hierarchically.

Let the linguistic term set be:

L = {hot, warm, cold}, where L is partially ordered as

$$\text{cold} \leq \text{warm} \leq \text{hot}.$$

Let us define a family of linguistic aggregation operators A_L, acting on pairs of inputs from L.

We define the following three operators:

- $a_{\inf}$: returns cold, which is the minimum of cold and hot;

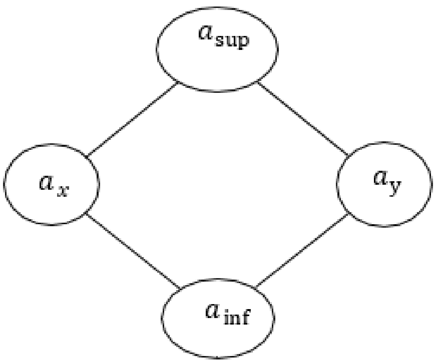

Fig. 3: Hasse diagram of the ordering relation defined on A.

- a_{avg}: returns warm, the middle value between cold and hot;
- $a_{\sup}$: returns hot, the maximum of cold and hot.

Thus, the aggregation family is:

$$A_L = \{a_{\inf}, a_{\text{avg}}, a_{\sup}\},$$

where $\leq$ is a partial order such that

$$a_{\inf} \leq a_{\text{avg}} \leq a_{\sup}.$$

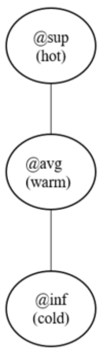

Fig. 4: Hasse diagram for the temperature example

Remark. The aggregation operator a_{avg} is defined so that, when the input linguistic terms are immediate successors with respect to the partial order, it returns the smaller term.
Fuzzy Linguistic Signatures (FLS) extend the concept of fuzzy Signatures (FSigs) by allowing the leaves of the signature tree to be assigned linguistic values or labels instead of fuzzy numbers. This structure enables the modeling of systems where qualitative, human-like reasoning is preferred over precise numerical values.

Definition 7: Fuzzy Linguistic Signatures

An FLS is represented as a tree linguistic structure G, where:

- Leaf nodes are assigned linguistic labels (e.g., low, medium, high).
- Internal nodes represent aggregation results of their child nodes using linguistic aggregation operators.

The formal definition of a fuzzy linguistic signature is

$$S_L = \langle N_I,\ N_L,\ \{a_1, \ldots, a_n\},\ \{L_1, L_2, \ldots, L_m\}\rangle.$$

Where:

S_L : Fuzzy linguistic signature,
N_I : Set of internal nodes of the graph G,
N_L : Set of leaf nodes of the graph G,
$V = N_I \cup N_L$ (Set of all vertices in G),
$\{a_1, \ldots, a_n\}$: Set of linguistic aggregation operators,
$\{L_1, L_2, \ldots, L_m\}$: Set of linguistic labels.

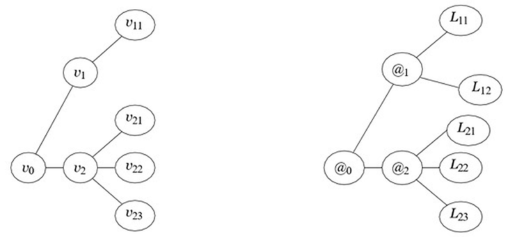

Fig. 5: Rooted tree G (left) and structure S_L (right)

An Example for Fuzzy Linguistic Signature Aggregation In the next, let us consider a very simple example. Let the traffic situation be described by the following set of linguistic expressions:

$$T_r = \big(\text{(Approximately) Zero},\ \text{Very Low},\ \text{Low},\ \text{Medium},\ \text{High},\ \text{Very High (Total Jam)}\big)$$

This set is linearly ordered, with $L_\downarrow$ = Zero and $L_\uparrow$ = Very High (see Figure 6. Let us assume now that the traffic intensity in a simple intersection is described by four-tuples of the traffic arriving from North, East, South, and West, denoted by

$$T_{r1} = (T_{rN}, T_{rE}, T_{rS}, T_{rW}),$$

a four-component vector-valued linguistic descriptor. This latter is analogous to and representable by a four-component vector-valued fuzzy descriptor [7]. However, in a real-life situation, the model is more adequate if the traffic is separately labeled in every possible incoming–outgoing direction, thus forming a subgraph with 12 leaves indexed by NW, NS, NE, EN, EW, etc.
For the simplest four-input example, if the traffic intensity situation is described by

$$T_r = (\text{Very Low, Low, High, Very High}),$$

where obviously, the EW and WE traffic is rather high, while the intersecting road has fewer vehicles approaching.
In the set of four-component linguistic traffic intensity descriptors, there exists a partial ordering defined based on the linear orderings of the component sets, namely:

$$T_{r1} \leq T_{r2} \Leftrightarrow$$
$$T_{rN1} \leq T_{rN2},\ T_{rE1} \leq T_{rE2},\ T_{rS1} \leq T_{rS2},\ T_{rW1} \leq T_{rW2}.$$

Here, $L_{\downarrow}$ = (Zero, Zero, Zero, Zero),
$L_{\uparrow}$ = (Very high, Very high, Very high, Very high).
The descriptor of a single intersection becomes more complex and requires a full fuzzy linguistic descriptor if, for example, for the intelligent traffic light control of the given intersection, a component describing the potential approach of one or more emergency vehicles is added. This necessitates a priority assignment to the given direction. Let us define the set of this additional feature by:

$$T_e = \big((\text{None},\ \text{Single},\ \text{Multiple},\quad \text{Many}\big)$$

Then the traffic intensity with added priority has two subtrees in every direction, both being single nodes describing T_e and T_r arriving from each direction. Figure 6 depicts the graph of such a simple linguistic signature describing a four-input intersection. Partial ordering can easily be defined here, too, as T_e is a linearly ordered set, and:

$$\begin{aligned}&(T_{r1}, T_{e1}) \leq (T_{r2}, T_{e2}) \ \Leftrightarrow\\ &\quad (T_{rN1}, T_{eN1}) \leq (T_{rN2}, T_{eN2}),\\ &\quad (T_{rE1}, T_{eE1}) \leq (T_{rE2}, T_{eE2}),\\ &\quad (T_{rS1}, T_{eS1}) \leq (T_{rS2}, T_{eS2}),\\ &\quad (T_{rW1}, T_{eW1}) \leq (T_{rW2}, T_{eW2}),\end{aligned}$$

$$\text{and } (T_{rJ1}, T_{eJ1}) \leq (T_{rJ2}, T_{eJ2}) \ \Leftrightarrow \begin{cases} T_{rJ1} \leq T_{rJ2},\\ T_{eJ1} \leq T_{eJ2}.\end{cases}$$

where J $\in \{N, E, S, W\}$.

Obviously, here:

$$\begin{aligned}L_{\downarrow} = \big(&(\text{Zero, None}),\ (\text{Zero, None}),\\ &(\text{Zero, None}),\ (\text{Zero, None})\big),\\ L_{\uparrow} = \big(&(\text{Jam, Many}),\ (\text{Jam, Many}),\\ &(\text{Jam, Many}),\ (\text{Jam, Many})\big).\end{aligned}$$

In [13], a simple traffic system was modeled, where traffic intensity and potential emergency vehicle appearance in the various directions, along with the waiting time of the queue (maybe consisting only of a single car), and the emergency vehicle priority in every incoming and outgoing direction, were taken into consideration. Let us define the following labels (which may be extended to all 12 directions):
W_t = {None, Short, Medium, Long, Very Long}, for the queues.
W_e = {None, Short, Long} , for potentially present emergency vehicles.
Here again, there is linear ordering in both sets, and there is a partial ordering in the set $W_t \times W_e$ so that $L_{\downarrow}$ = (None, None) and $L_{\uparrow}$ = (Very Long, Long).

It depends on the application of a given FLS and how the aggregations of the model structure can be defined. In [13], a fuzzy rule-based strategy was proposed, and in this illustrative example, a similar but more compact model and algorithm may be constructed based on FLS as above. The real-life application of such intelligent traffic control algorithms becomes interesting when a (maybe rather large) system of interconnected intersections is modeled as a single object. Traffic itself generates the information flow connecting the whole system, and it may be expected that some general features or patterns in the behavior of the traffic lights emerge as the result of the entire system adapting to the input "signals," namely, the traffic intensities and emergency vehicle appearances in the intersection system.

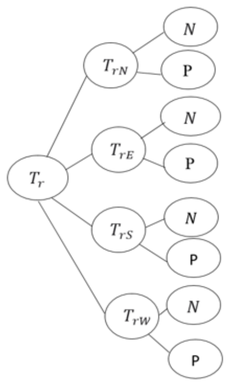

Fig. 6: Fuzzy linguistic tree for the traffic example

In the above paragraphs, it was presented how the graph structure of an FLS can be built up, and how the leaves of the trees can be linguistic values assigned. However, the problem of aggregations in the intermediate nodes (including the root itself) is just as important when the structure of the FLS is determined.

Let us continue the above example and focus on the traffic light control signal for the NS direction in a given intersection. The control output may be:

- G = Switch immediately to/keep on Green,
- GD = Switch with Delta delay to/keep on Green,
- RD = Switch with Delta delay to/keep on Red,
- R = Switch immediately to/keep on Red.

Then, in the simplest four-direction example, the linguistic aggregation for the FLS determining the traffic light of the North incoming street may be constructed as follows:

- The traffic light must be set to G if

$$(T_{rN}, T_{eN}) \geq \max\{(T_{rE}, T_{eE}),\ (T_{rW}, T_{eW})\}.$$

In a refined model, (T_{rS}, T_{rN}) is also taken into consideration.

The aggregation

$$A_N = A_N(T_{rN}, T_{eN}, T_{rE}, T_{eE}, T_{rW}, T_{eW}, W_{tE}, W_{tW})$$

(maybe also including T_{rS} and T_{eS}) should be defined by a lookup table, where the size grows exponentially with the number of inputs. Thus, the vector-valued approach is not feasible; instead, a real (multiple-level) FLS should be constructed with subtrees representing the "strength" or degree of each direction pushing the control decision towards a given value.

III. DEFINITION OF MEET AND JOIN FOR FUZZY LINGUISTIC SIGNATURES

Fuzzy Linguistic Signatures (FLS) are designed to represent uncertain and imprecise data in situations where exact fuzzy membership functions and degrees cannot be precisely specified. As an extension of fuzzy signatures (FSigs), it is reasonable to maintain FLS within a similar mathematical framework as FSigs. Previous research has shown that FSigs form a lattice structure [1], so it would be valuable to explore whether FLSs exhibit the same behavior, under what conditions this occurs, and to examine all potential cases. To demonstrate that FLSs form a lattice, we will discuss the meet and join operators and investigate if FLS satisfies the properties of idempotency, commutativity, and associativity. This proof will build on the findings in [1], while taking into account the unique requirements of linguistic labels. We further assume that the sets of linguistic labels and linguistic aggregation operators form partially ordered sets that admit infimum and supremum. For any given application, it is necessary that the set of linguistic labels contains a supremum and an infimum element, which semantically and intuitively bound all other linguistic labels from above and below, respectively.

Definition 8: Fuzzy Linguistic Signatures Generated from G and A_L

Let $G = (V, E)$ be a tree with root v_0, whose set of internal vertices is given by $N_I = \{v_0, v_1, v_2, \dots, v_n\}$, and let $A_L = \{A_{L0}, A_{L1}, A_{L2}, \dots, A_{Ln}\}$, be a set of families of aggregation operators. The family of fuzzy linguistic Signatures generated from G and A_L, which will be denoted as $F(G, A_L)$, is defined as follows:

$$F = \Big\{ S_{Lk} = \langle N_{Ik},\, N_{Lk},\, \{a_{i1}, \dots, a_{ip}\}, L_1, \dots, L_q\rangle \;\Big|\; G_k \subseteq G,\; a_{ij} \in A_{Lij},\; \text{for all } v_{ij} \in N_{Ik} \Big\}.$$

where $G_k = (N_{Ik} \cup N_{Lk}, E_k)$ is a subgraph of G, satisfying that $v_0 \in N_{Ik}$, and S_{Lk} is a fuzzy linguistic signature associated with G_k.

We are also interested in defining the family of fuzzy linguistic signatures generated from a given fuzzy signature. Notice that such a concept of family will be suitable for formalizing the information corresponding to situations similar to the ones explained at the beginning of this section.

Definition 9: The Family of Fuzzy Linguistic Signatures Generated from S_L

Let S_L be a fuzzy linguistic signature associated with the rooted tree G_s and the set of families of aggregation operators A_s. The family of fuzzy linguistic signatures generated from S_L is defined as the family of fuzzy linguistic signatures $F(G_s, A_s)$.

Therefore, given a fuzzy linguistic signature S_L, the family $F(G_s, A_s)$ is the set of fuzzy linguistic signatures S'_L obtained from S_L by omitting any number of leaves, or leaf subtrees (except the root itself), with the necessary corresponding modifications to maintain the definition of fuzzy linguistic signature. The labels of the internal vertices v of S'_L are associated with aggregation operators in the same family as the label of this vertex v in S_L. When a whole subtree has been removed, the root of this subtree becomes a leaf instead of its original non-leaf position in the original fuzzy linguistic signature and so, a linguistic label will be assigned to it, instead of a linguistic aggregation operator.

Definition 10 (Join of the Fuzzy Linguistic Signatures).

Let S_L be a fuzzy linguistic signature associated with the rooted tree G_s and the family of linguistic aggregation operators A_s. Let $S_{L_1}, S_{L_2} \in \mathcal{F}(G_s, A_s)$ be fuzzy linguistic signatures associated with the rooted trees $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where N_{L_1}, N_{L_2} denote the sets of leaves and N_{I_1}, N_{I_2} denote the sets of internal vertices of G_1 and G_2.

The join of the fuzzy linguistic signatures S_{L_1} and S_{L_2}, denoted by $S_{L_1} \cup S_{L_2}$, is the fuzzy linguistic signature associated with the rooted tree

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2),$$

where $N_I(G_1 \cup G_2)$ denotes the set of internal vertices and $N_L(G_1 \cup G_2)$ denotes the set of leaves.

The linguistic aggregation operator assigned to each internal vertex $v \in N_I(G_1 \cup G_2)$ is defined by the following cases:

1. Internal vertices (Linguistic aggregation operators).

1) If $v \in N_{I_1}$ and $v \in N_{I_2}$, then
$$a_v = \sup\{a_v^1, a_v^2\},$$
where a_v^1 and a_v^2 are the linguistic aggregation operators assigned to v in S_{L_1} and S_{L_2}, respectively.
2) If $v \in N_{I_1}$ and $v \notin V_2$, then
$$a_v = a_v^1.$$
3) If $v \notin V_1$ and $v \in N_{I_2}$, then
$$a_v = a_v^2.$$
4) If $v \in N_{I_1}$ and $v \in N_{L_2}$, then
$$a_v = a_v^1.$$
5) If $v \in N_{L_1}$ and $v \in N_{I_2}$, then
$$a_v = a_v^2.$$

Remark. Since S_{L_1} and S_{L_2} belong to the same family of fuzzy linguistic signatures, their structures may not be identical. Consequently, it may occur that a given vertex is not present in one of the two signatures, or that a given vertex is a leaf in one signature and an internal vertex in the other. The above rules explicitly define how the join operation is computed in all these cases.

2. Leaves (Linguistic Labels):

If $v \in N_L\{G_1 \cup G_2\}$ (a leaf), the linguistic label assigned to v is computed considering the following cases:

1) If $v \in N_{L1}$ and $v \in N_{L2}$, the linguistic label assigned to v is:
$$L_v = \sup\{L_v^1, L_v^2\}$$
where L_v^1 is the linguistic label assigned to $v \in N_{L1}$ and L_v^2 is the linguistic label assigned to $v \in N_{L2}$.
2) If $v \in N_{L1}$ and $v \notin N_{L2}$, the linguistic label assigned to v is $L_v = L_v^1$, where L_v^1 is the linguistic label assigned to $v \in N_{L1}$.
3) If $v \notin N_{L1}$ and $v \in N_{L2}$, the linguistic label assigned to v is $L_v = L_v^2$, where L_v^2 is the linguistic label assigned to $v \in N_{L2}$.

Mixed cases in which a vertex is internal in one fuzzy linguistic signature and a leaf in the other are resolved according to the convention stated in the internal-vertex cases above.

Definition 11: Meet of the Fuzzy Linguistic Signatures

Let S_L be a fuzzy linguistic signature associated with the rooted tree G_s and the family of linguistic aggregation operators A_s. Let $S_{L_1}, S_{L_2} \in \mathcal{F}(G_s, A_s)$ be fuzzy linguistic signatures associated with the rooted trees $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where N_{L_1}, N_{L_2} denote the sets of leaves and N_{I_1}, N_{I_2} denote the sets of internal vertices of G_1 and G_2. For each tree G_i, the vertex set satisfies $V_i = N_{L_i} \cup N_{I_i}$.

The meet of the fuzzy linguistic signatures S_{L_1} and S_{L_2}, denoted by $S_{L_1} \cap S_{L_2}$, is the fuzzy linguistic signature associated with the rooted tree
$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2),$$
where $N_I(G_1 \cap G_2)$ denotes the set of internal vertices and $N_L(G_1 \cap G_2)$ denotes the set of leaves.

The linguistic aggregation operator assigned to each vertex in $N_I(G_1 \cap G_2)$ and the linguistic label assigned to each vertex in $N_L(G_1 \cap G_2)$ are defined as follows:

1. Internal vertices (Linguistic aggregation operators).

If $v \in N_I(G_1 \cap G_2)$, then the linguistic aggregation operator assigned to v is
$$a_v = \inf\{a_v^1, a_v^2\},$$
where a_v^1 and a_v^2 are the linguistic aggregation operators assigned to v in S_{L_1} and S_{L_2}, respectively.

The treatment of vertices that are internal in one fuzzy linguistic signature and leaves in the other follows the same convention as described in the remark following the definition of the join operation.

2. Leaves (Linguistic labels).

If $v \in N_L(G_1 \cap G_2)$, the linguistic label assigned to v is defined by the following cases:

1) If $v \in N_{L_1}$ and $v \in N_{L_2}$, then
$$L_v = \inf\{L_v^1, L_v^2\},$$
where L_v^1 and L_v^2 are the linguistic labels assigned to v in S_{L_1} and S_{L_2}, respectively.
2) If $v \in N_{L_1}$ and $v \notin N_{L_2}$, then
$$L_v = L_v^1.$$
3) If $v \notin N_{L_1}$ and $v \in N_{L_2}$, then
$$L_v = L_v^2.$$

IV. Lattice Properties of Fuzzy Linguistic Signatures

A. Structural Compatibility Assumption

Throughout this section, we restrict our attention to the family $\mathcal{F}(G_s, A_s)$ of fuzzy linguistic signatures defined over an identical rooted tree $G_s = (V_s, E_s)$. Consequently, for any two signatures $S_{L_1}, S_{L_2} \in \mathcal{F}(G_s, A_s)$, each vertex $v \in V_s$ is either a leaf in both signatures or an internal vertex in both. Thus, mixed cases where a vertex is a leaf in one signature and an internal vertex in the other do not occur.

We further assume that the sets of linguistic labels and linguistic aggregation operators are partially ordered and admit infimum and supremum. Under these assumptions, the meet ($\cap$) and join ($\cup$) operations defined in Definitions 10 and 11 are well defined and operate node-wise on G_s. If the underlying rooted trees are not identical, the proofs can be formulated in a very similar manner by a case-by-case analysis, following the constructions given in Definitions 10 and 11.

B. Lattice Properties of Fuzzy Linguistic Signatures

Let $S_{L_1}, S_{L_2}, S_{L_3} \in \mathcal{F}(G_s, A_s)$. We verify that the structure
$$(\mathcal{F}(G_s, A_s), \cap, \cup)$$
satisfies the lattice axioms.

a) Commutativity.:
$$S_{L_1} \cap S_{L_2} = S_{L_2} \cap S_{L_1}, \qquad S_{L_1} \cup S_{L_2} = S_{L_2} \cup S_{L_1}.$$

Proof. By Definition 11, for any internal vertex $v \in N_I(G_s)$,
$$a_v = \inf\{a_v^1, a_v^2\} = \inf\{a_v^2, a_v^1\},$$
and for any leaf $v \in N_L(G_s)$,
$$L_v = \inf\{L_v^1, L_v^2\} = \inf\{L_v^2, L_v^1\}.$$
Thus, the resulting aggregation operators and linguistic labels are independent of the order of the operands. The same argument applies to the join operation using the supremum. *End of proof.*

b) Associativity.:

$$(S_{L_1} \cap S_{L_2}) \cap S_{L_3} = S_{L_1} \cap (S_{L_2} \cap S_{L_3}),$$

$$(S_{L_1} \cup S_{L_2}) \cup S_{L_3} = S_{L_1} \cup (S_{L_2} \cup S_{L_3}).$$

Proof. For any internal vertex $v \in N_I(G_s)$,

$$\inf\{\inf\{a_v^1, a_v^2\}, a_v^3\} = \inf\{a_v^1, \inf\{a_v^2, a_v^3\}\},$$

by associativity of the infimum. Similarly, for any leaf $v \in N_L(G_s)$,

$$\inf\{\inf\{L_v^1, L_v^2\}, L_v^3\} = \inf\{L_v^1, \inf\{L_v^2, L_v^3\}\}.$$

Hence, aggregation results coincide at every vertex. The same reasoning applies to the join operation using supremum. *End of proof.*

c) Idempotency.:

$$S_{L_1} \cap S_{L_1} = S_{L_1}, \qquad S_{L_1} \cup S_{L_1} = S_{L_1}.$$

Proof. For any internal vertex $v \in N_I(G_s)$,

$$\inf\{a_v^1, a_v^1\} = a_v^1,$$

and for any leaf $v \in N_L(G_s)$,

$$\inf\{L_v^1, L_v^1\} = L_v^1.$$

Thus, applying meet or join to a signature with itself leaves it unchanged. *End of proof.*

d) Absorption.:

$$S_{L_1} \cap (S_{L_1} \cup S_{L_2}) = S_{L_1}, \qquad S_{L_1} \cup (S_{L_1} \cap S_{L_2}) = S_{L_1}.$$

Proof. For any internal vertex $v \in N_I(G_s)$,

$$\inf\{a_v^1, \sup\{a_v^1, a_v^2\}\} = a_v^1,$$

and for any leaf $v \in N_L(G_s)$,

$$\inf\{L_v^1, \sup\{L_v^1, L_v^2\}\} = L_v^1,$$

by the absorption property of infimum and supremum. The dual equality follows analogously. *End of proof.*

C. Resulting Lattice Structure of Fuzzy Linguistic Signatures

Since the meet and join operations satisfy commutativity, associativity, idempotency, and absorption, the structure

$$(\mathcal{F}(G_s, A_s), \cap, \cup)$$

forms a lattice of fuzzy linguistic signatures.

V. CONCLUSION

Linguistic variables are very useful in describing and handling everyday situations, especially those involving subjectivity and the terminology of natural language. When linguistic features can be structured in a multi-component hierarchy, this motivates the introduction of fuzzy linguistic signatures (FLS), which are particularly important for modeling such subjective cases.

FLS are similar in structure to fuzzy signatures. However, instead of fuzzy membership degrees and fuzzy membership functions, linguistic values are applied on the leaves. In the internal nodes, linguistic aggregation must appear; thus, we define the concept of linguistic aggregation.

We have shown that FLS can be handled if they belong to the same family of FLS that can be derived from a fuzzy linguistic mother signature by omission of the edges and the corresponding nodes. Such FLS belonging to the same family can be compared and aggregated.

We have shown that FLS of a family form a lattice, and in this way, they can be considered as an extension of the concept of L-fuzzy sets.

We have shown an example of decision-making in the context of a traffic situation, and we have presented that linguistic variables and FLS are suitable for modeling and making decisions in this application area.

REFERENCES

[1] L. T. Kóczy, M. E. Cornejo, and G. Medina, "Algebraic structure of fuzzy signatures," *Fuzzy Sets and Systems*, vol. 418, pp. 25–50, 2021, **DOI**: 10.1016/j.fss.2020.09.020.

[2] K. W. Wong, T. D. Gedeon, and L. T. Kóczy, "Fuzzy signature and cognitive modelling for complex decision model," in *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing*, O. Castillo, P. Melin, O. M. Ross, R. Sepúlveda Cruz, W. Pedrycz, and J. Kacprzyk, Eds., ser. Advances in Soft Computing, vol. 42, Springer, 2007, pp. 380–389, **DOI**: 10.1007/978-3-540-72434-6_39.

[3] G. Mikulás and L. T. Kóczy, "Macro-level road network evaluation by fuzzy signature rule bases," *Hungarian Statistical Review: Journal of the Hungarian Central Statistical Office*, vol. 4, no. 1, pp. 3–16, 2021.

[4] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning–I," *Information Sciences*, vol. 8, pp. 199–249, 1975, **DOI**: 10.1016/0020-0255(75)90036-5.

[5] L. T. Kóczy, "Vector valued fuzzy set," *BUSEFAL (Université Paul Sabatier, Toulouse)*, pp. 41–57, 1980.

[6] Z. Xu, "Linguistic aggregation operators: An overview," in *Fuzzy Sets and Their Extensions: Representation, Aggregation and Models*, ser. Studies in Fuzziness and Soft Computing, vol. 220, Springer, 2007, pp. 163–181, **DOI**: 10.1007/978-3-540-72434-6_7.

[7] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar, "Aggregation operators: Properties, classes and construction methods," in *Aggregation Operators, ser. Studies in Fuzziness and Soft Computing*, vol. 97, Heidelberg: Physica, 2002, pp. 3–104, **DOI**: 10.1007/978-3-7908-1781-1_1.

[8] F. Herrera and J. L. Verdegay, "Linguistic assessments in group decision," in *Proc. 1st European Congress on Fuzzy and Intelligent Technologies*, Aachen, 1993, pp. 941–948.

[9] F. Herrera and L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 8, pp. 746–752, 2000, **DOI**: 10.1109/91.890332.

[10] Z. Xu, "On generalized induced linguistic aggregation operators," *International Journal of General Systems*, vol. 35, pp. 17–28, 2006, **DOI**: 10.1080/03081070500226824.

[11] R. R. Yager, "On the ordered weighted averaging operators in multicriteria decision making," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 18, pp. 183–199, 1988, **DOI**: 10.1109/21.87068.

[12] T. D. Chala and L. T. Kóczy, "Intelligent fuzzy traffic signal control system for complex intersections using fuzzy rule base reduction," *Symmetry*, vol. 16, no. 9, p. 1177, 2024, **DOI**: 10.3390/sym16091177.

[13] T. D. Chala and L. T. Kóczy, "A novel, three-stage intelligent fuzzy traffic signal control system," *Acta Polytechnica Hungarica*, vol. 21, no. 8, 2024.

[14] N. Ammar and L. T. Kóczy, "Decision-Making Based on Fuzzy Linguistic Signatures," in *Proc. 4th Int. Conf. on Communications, Information, Electronic and Energy Systems (CIEES)*, Nov. 2023, pp. 1–5.

[15] L. A. Zadeh, "Fuzzy logic = computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 2, pp. 103–111, May 1996, **DOI**: 10.1109/91.493904.

[16] N. Ammar and L. T. Kóczy, "Fuzzy Linguistic Signatures and Their Applications," in *Proc. 28th IEEE Int. Conf. on Intelligent Engineering Systems (INES)*, July 2024, pp. 27–30.

[17] Z. H. Radhy, F. H. Maghool, and N. K. Hady, "Fuzzy-assignment model by using linguistic variables," *Baghdad Science Journal*, vol. 18, no. 3, p. 5, 2021.

[18] D. Liu, Y. Liu, and X. Chen, "Fermatean fuzzy linguistic set and its application in multicriteria decision making," *International Journal of Intelligent Systems*, vol. 34, no. 5, pp. 878–894, 2019, **DOI**: 10.1002/int.22079.

Nour Ammar is a PhD candidate at the Department of Telecommunications and Artificial Intelligence, Budapest University of Technology and Economics, Hungary, where she also obtained her Master's degree. Her research focuses on fuzzy systems and fuzzy linguistic signatures, particularly their role in decision-making.

László T. Kóczy is a Professor at the De- partment of Information Technology, Széchenyi István University; Foreign Member of the Polish Academy of Sciences; Fellow of the International Fuzzy Systems Association; LIFE Honorary President of the Hungarian Fuzzy Association.