On the Convex Hull of the Achievable Capacity Region of the Two User FDM OMA Downlink

Zoltán Belső and László Pap

Abstract—**In multiple access channel systems, such as a mobile communication network, it is important to determine how to share the available resources (for example bandwidth and power) among the users. In recent years, one of the promising scheme is Non-Orthogonal Multiple Access (NOMA), where, unlike the traditional Orthogonal Multiple Access, OMA solution, signals for the different users overlap in some domain (power-domain NOMA, code-domain NOMA, etc). In order to evaluate the performance of any NOMA scheme, we need to compare the achievable bit rates of the users (the capacity region) to an OMA case with comparable parameters (for example, same total bandwidth, same total power and same channel conditions, etc). To make this comparison, we first need to know the capacity region for the OMA cases. Many papers make such comparison without detailing the derivation of the capacity region of the OMA case they compare to [1], [2], [3], [4]. In some cases, we have only one free parameter to choose (for example in uplink frequency division multiplexing systems for two users, it is the bandwidth ratio between the users), and the achievable capacity can be directly calculated for both users depending on the single parameter (hence the boundary of the capacity region is trivial). In other cases, such as downlink frequency division multiplexing systems, even for only two users, we have to allocate optimally two resources between the users: the bandwidth and the base station's available power. Hence, it is far from being trivial to determine which combination is better and where the boundary of the capacity region is. In this paper, we provide a derivation for that case.**

Index Terms—**Orthogonal Multiple Access (OMA); Non-Orthogonal Multiple Access (NOMA); capacity region**

I. Introduction

In the mobile communication networks, where a base station is communicating with multiple stations, there are two different cases: uplink and downlink communication, each with somewhat different problems that need to be solved. In the classical orthogonal multiple access (OMA) system, which utilizes frequency division multiple access for communication, there are two resources to share between the users: the bandwidth and the available transmission power.

In the uplink communication case, where the end users attempt to transmit signal to the base station, all the users have their own transmit power limits. Since the communication is orthogonal, the maximum capacity is reached when all the users transmit at maximum transmission power. The only resource that needs to be shared between the users is the available bandwidth. In the case of two users, we can allocate some portion, denoted by $\alpha \in [0,1]$, of the whole available bandwidth to one of the users and the remainder to the other. This allocation determines the boundary of the achievable capacity region for both users.

In the downlink communication case, where the base station transmits to many users, the power budget of the base station also has to be split between the users, alongside the available bandwidth. So, in the case of two users, we have two parameters: the bandwidth division ratio $\alpha \in [0,1]$, and the power division ratio $\beta \in [0,1]$; both can be chosen independently on a two-dimensional capacity plane. Any choice of these two parameters gives us a capacity limit for both users (with a given channel characteristics) that can be represented on a capacity plane, where one axis is the capacity of the first user and the other axis is the capacity of the other user. The convex hull of these points (capacity pairs) gives us the boundary of the achievable capacity region.

In order to examine the shape of the achievable capacity region on the capacity plane we can fix one of these two parameters and let the other changing within its range: this way we get a curve on the capacity plane. We can calculate these curves for any value of the first parameter. This way we get a curve array, and the convex hull of them also gives the same boundary of the achievable capacity region. One can easily see that most of the parameter value pairs are suboptimal. For example, allocating all the power to one user while not allocating all the bandwidth at the same time cannot be optimal: the second user has 0 capacity, and we can increase the first user's capacity by allocating more power to them without any further decreasing the capacity of the second user. So without any complicated derivation we can say that the optimal corner case is that while allocating all the power to one of the users all the bandwidth must be allocated to the same user. Other cases are not trivial. One guess could be to allocate the same portion of power and bandwidth to the users is the optimal allocation, but we will later see that this is only the case when the channel conditions of the two users are the same (it is easy to see that for that case it is optimal, due to symmetry reasons), but in any other cases there are better allocations. Some textbooks (see [5] fig. 15.29, [6] fig. 6.9) gives us the optimal curve, but without derivation or literature reference.

We can get a sense of this capacity region by plotting a school of curves on a capacity plane by choosing a value for one of the parameters (in this case the power division ratio, $\beta \in [0,1]$) and run the other (in this case the bandwidth ratio, $\alpha \in [0,1]$) through the whole range. We can see this curves as a kind of iso-power-ratio lines. The convex hull of all possible curves gives us the capacity region. See Figure 1.

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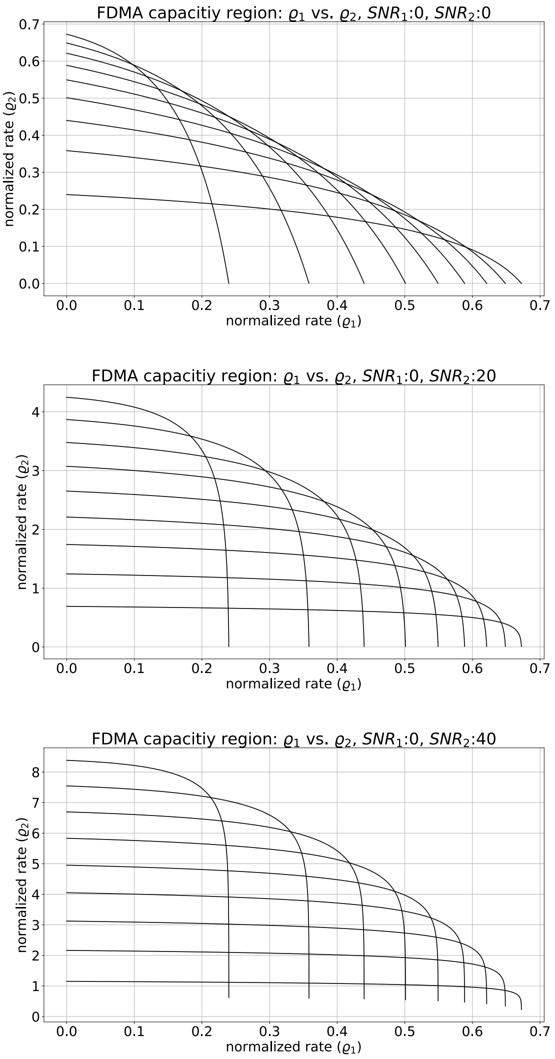

Figure 1. The rate pairs for different bandwidth and power distribution, and the limit of the capacity region

Note, that only one point of each curve falls onto the convex hull, and in the case of differing signal-to-noise ratio the boundary is clearly not a straight line (that would correspond to the case when we chose the same value for both of the resource allocation parameters: $\alpha = \beta$). In some literature (like [4] fig. 4.) the capacity region boundary for the OMA case is graphed as a linear function even in the case of different receiving signal-to-noise ratio (SNR) conditions for the users, which is not correct. There are literature (like [2] fig. 2.) that shows a seemingly correct convex curve as the capacity region of the OMA case, but without derivation or reference. Some others show graphs that are simplified enough not to be conclusive [3].

One can also ask a broader question of what about having more than two users. In this case the situation is even more complicated since we have to split the available resources between all the users. Let's denote the number of users by N, then we have to choose $\alpha_1, \alpha_2 \dots \alpha_N \in [0,1]$ and $\beta_1, \beta_2 \dots \beta_N \in [0,1]$ such that $\sum_{i=1}^{N} \alpha_i = 1$ and $\sum_{i=1}^{N} \beta_i = 1$. So we have an $N \times N$ parameter space and find the optimal combination of these parameters to get the convex hull of the achievable capacity region. In this paper, we will concentrate on the two user case.

In this paper, we give a derivation of the curve by expressing the capacity for the second user as a function of the capacity of the first user and one of the two parameters, namely α, and finding the extreme value (maximum) of these function with respect of the free parameter α, while keeping the capacity limit of the first user fixed. This way, we can determine the maximum capacity that the second user can reach for any given capacity of the first user. We can calculate and plot the limits of the capacity region for different channel conditions of the users.

This paper is organized as follows: First, we provide a formulation of the problem by expressing the normalized capacity of both users with respect of the two free parameters. Then, we express the function for which we need to find the extreme value. After that, we express the limits of the parameters which are applicable for any given capacity of the first user (which we will consider fixed), and examine the function to maximize by giving the graph of it. Next, we express the derivative of the function with respect of the free parameter and find the zero crossing of the derivative. Finally, we plot the found capacity limit pairs of the users. We close the paper with a conclusion section.

II. Problem formulation

Let's consider the downlink case (OMA) with frequency division multiplexing. In this scenario, we have a base station with a maximal transmitted power limit and an available total bandwidth. Let's consider that we have two users and the base station wants to transmit signal to both users simultaneously.

Here we have two free parameters that can be split arbitrarily between the two users:

- We have to divide the available bandwidth, let's denote the division ratio by $\alpha \in [0,1]$
- We also have to divide the transmitted power of the base station independently. Let's denote the division ratio by $\beta \in [0,1]$

The two users are located in a different physical location, so the channel characteristic may differ for the two users. Let's denote the complex channel characteristic for the two users by $h_i \; i \in (1,2)$.

This way, the achievable bit rate is as follows:[5], [6]

$$\begin{aligned} R_1 &= \alpha W \log_2 \left(1 + \frac{\beta P \left| h_1 \right|^2}{\alpha W N_0} \right) \\ R_2 &= (1-\alpha) W \log_2 \left(1 + \frac{(1-\beta) P \left| h_2 \right|^2}{(1-\alpha) W N_0} \right) \end{aligned} \tag{1}$$

Where W denotes the total bandwidth available, with αW and $(1-\alpha)W$ representing the allocated bandwidth to user 1 and 2, respectively, P denotes the total transmission power of the base station, with βP and $(1-\beta)P$ are the allocated transmission power to user 1 and 2, respectively. Without loosing generality, we consider that the environmental parameter N_0, the noise spectral power density, is the same for both users.

Let's normalize the bandwidth to unit Hertz and get rid of the $\log_2()$ by multiplying with the constant $\ln(2)$. Also divide both the numerator and the denominator of the fraction inside the parentheses representing the signal-to-noise ratio of the users by W, so we use the total transmission energy $E = \frac{P}{W}$ instead of the total transmission power P:

$$\begin{aligned} \varrho_1 =& R_1 \frac{\ln(2)}{W} \\ =& \alpha \ln\left(1 + \frac{\beta}{\alpha}\frac{E\,|h_1|^2}{N_0}\right) \\ \varrho_2 =& R_2 \frac{\ln(2)}{W} \\ =& (1-\alpha)\ln\left(1 + \frac{(1-\beta)}{(1-\alpha)}\frac{E\,|h_2|^2}{N_0}\right) \end{aligned} \tag{2}$$

Let's further denote the best possible (when all the resources allocated to that user) signal-to-noise ratios for each user as:

$$\begin{aligned} A_1 &= \frac{E\,|h_1|^2}{N_0} \\ A_2 &= \frac{E\,|h_2|^2}{N_0} \end{aligned} \tag{3}$$

Which gives the expressions for ϱ_1 and ϱ_2 in the following form:

$$\begin{aligned} \varrho_1 =& \alpha \ln\left(1 + \frac{\beta}{\alpha} A_1\right) \\ \varrho_2 =& (1-\alpha)\ln\left(1 + \frac{(1-\beta)}{(1-\alpha)} A_2\right) \end{aligned} \tag{4}$$

The question can be formulated as follows: Given the value of ϱ_1, what is the maximal value of ϱ_2?

III. The function to be maximized

We have two free parameters α and β, but we don't want to choose them independently, since those would pin down both ϱ_1 and ϱ_2. Instead, we choose one of the rate (without giving up generality we can choose ϱ_1), so we can derive an expression between the parameters, and we can eliminate one of them (again without giving up generality we can choose β to eliminate). That way, we can search for the maximum of ϱ_2 as a function of α for a given ϱ_1.

For a given α and ϱ_1 value, we can express β as:

$$\exp\left(\frac{\varrho_1}{\alpha}\right) = 1 + \frac{\beta}{\alpha} A_1 \tag{5}$$

$$\beta = \alpha \frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1} \tag{6}$$

We got a closed expression for β for a given α and ϱ_1, which we can substitute into the expression for ϱ_2 in (4):

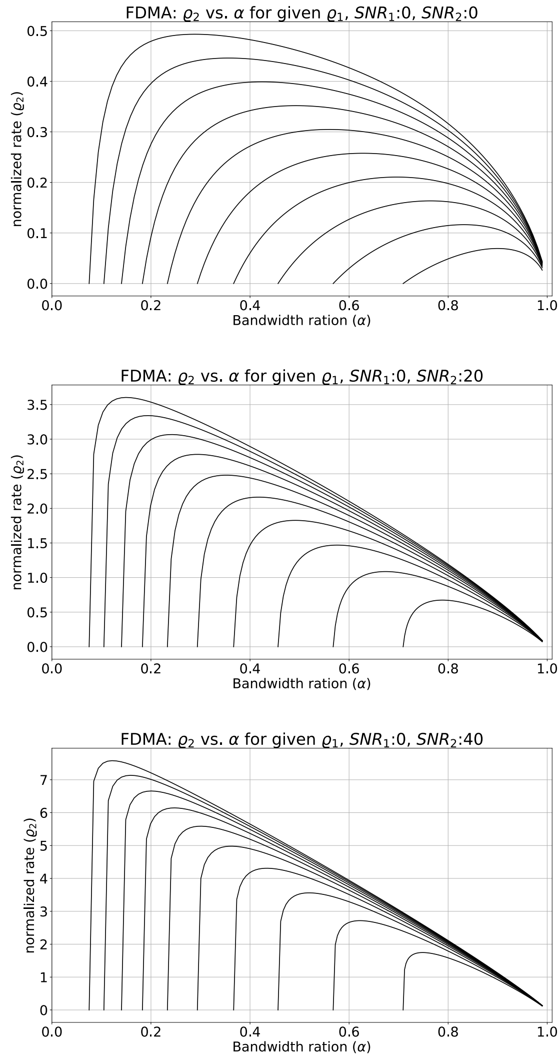

Figure 2. Achievable rate of user2 versus the bandwidth division ratio for a rate of user1. The normalized rate of user1 for the curves from left to right are: $\varrho_1 = 0.10, 0.16, 0.22, 0.33, 0.39, 0.45, 0.51, 0.57, 0.62$

$$\varrho_2(\alpha) = (1-\alpha)\ln\left(1 + \left(1 - \alpha\frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1}\right)\frac{A_2}{1-\alpha}\right) \tag{7}$$

This is the function for which we are looking for an extreme value, with respect of α. We are looking for such a maximum value for any given ϱ_1 value. That way, we get a curve on the two-dimensional ϱ_1, ϱ_2 plane.

One can see this optimization problem as follows: There is a range of achievable capacity for one user, that is between 0 and the maximum that can be reached when the total bandwidth and power allocated to that user. Any capacity in between these limits can be achieved by some (possible many) combinations of the free parameters α and β. The question can be asked is: what is the maximum capacity achievable by the other user for any given rate of the first user?

IV. EXAMINING THE RATE FUNCTION

In order to graph the rate function in (7) we need to find the limits of the domain α.

For the domain of meaningful α values, we can consider the limitations from the original expressions:

Since both α and β means a division ratio:

$$0 \leq \quad \alpha \quad \leq 1 \tag{8}$$
$$0 \leq \quad \beta \quad \leq 1 \tag{9}$$

Also, the rates ϱ_1 and ϱ_2 must be a positive value.

We can expand (9) utilizing the expression for β in (6):

$$0 \leq \quad \alpha \frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1} \quad \leq 1 \tag{10}$$
$$0 \leq \quad \exp\left(\frac{\varrho_1}{\alpha}\right) - 1 \quad \leq \frac{A_1}{\alpha} \tag{11}$$
$$1 \leq \quad \exp\left(\frac{\varrho_1}{\alpha}\right) \quad \leq 1 + \frac{A_1}{\alpha} \tag{12}$$
$$0 \leq \quad \frac{\varrho_1}{\alpha} \quad \leq \ln\left(1 + \frac{A_1}{\alpha}\right) \tag{13}$$
$$0 \leq \quad \varrho_1 \quad \leq \alpha \ln\left(1 + \frac{A_1}{\alpha}\right) \tag{14}$$

This is given as a (greater than 0) lower limit for α. The upper limit is $\alpha \leq 1$:

$$\frac{\varrho_1}{\ln\left(1 + \frac{A_1}{\alpha}\right)} \leq \quad \alpha \quad \leq 1 \tag{15}$$

As one can see, it is unfortunately a transcendent inequality in terms of α, so it doesn't give us a limit for α in a closed form. But we can find the smallest α value, that gives $\varrho_1 \geq 0$ by solving numerically for:

$$\frac{\varrho_1}{\ln\left(1 + \frac{A_1}{\alpha}\right)} - \alpha = 0 \tag{16}$$

Solving the limits (16) numerically, we can graph the function from (7) for different ϱ_1 values for some SNR cases, see Figure 2. Examining these curves, one can see that there is a unique maximum point for the ϱ_2 normalized rate for user2 and a corresponding α parameter value for any given ϱ_1 normalized rate of user1.

V. FINDING THE EXTREME VALUE

Let's find the maximum of the expression in (7) with respect to α, that is, calculating $\frac{\partial \varrho_2}{\partial \alpha}$ considering ϱ_1 as a constant parameter during the differentiation. We can find the extreme value by solving for $\frac{\partial \varrho_2}{\partial \alpha} = 0$ and we can do that for all the possible values of ϱ_1.

Since (7) contains a composite function, an $\ln$ with a rather complicated inner expression, we will have to calculate

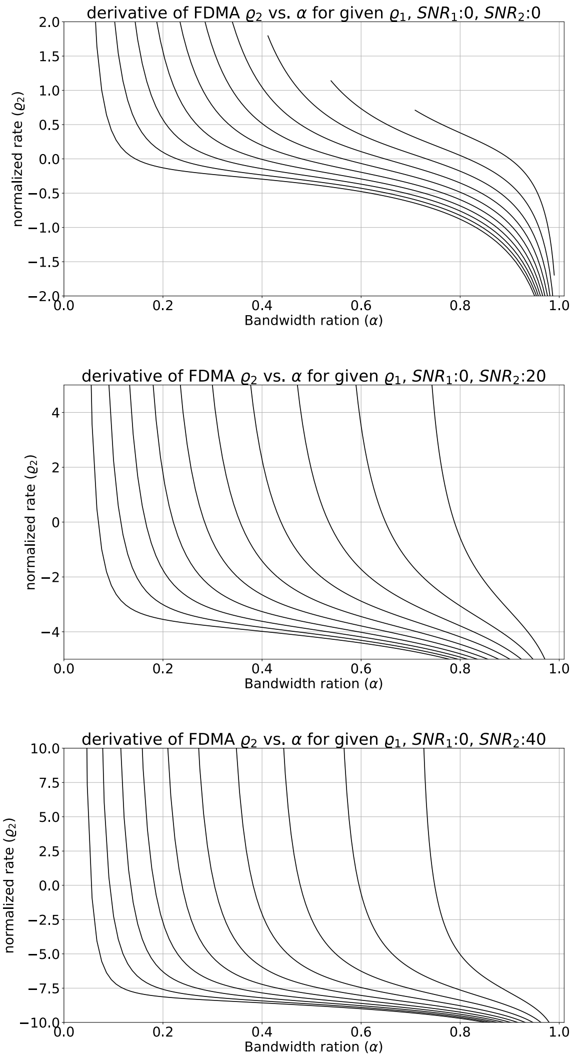

Figure 3. Derivative of achievable rate of user2 versus the bandwidth division ratio for different rates of user1. The normalized rate of user1 for the curve from left to right are: $\varrho_1 = 0.10, 0.16, 0.22, 0.33, 0.39, 0.45, 0.51, 0.57, 0.62$

the derivative of the expression inside the $\ln$ function, let's calculate that first:

$$\begin{aligned} &\frac{\partial}{\partial \alpha}\left(1 + \left(1 - \alpha \frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1}\right) \frac{A_2}{1-\alpha}\right) = \\ &= \left(-\frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1} + \frac{\alpha}{A_1} \exp\left(\frac{\varrho_1}{\alpha}\right) \frac{\varrho_1}{\alpha^2}\right) \frac{A_2}{1-\alpha} + \\ &- \left(1 - \alpha \frac{\exp\left(\frac{\varrho_1}{\alpha}\right) - 1}{A_1}\right) \frac{-A_2}{(1-\alpha)^2} \end{aligned} \tag{17}$$

Using (17) we can utilize the chain rule and write the derivative of the function in (7) as:

$$\frac{\partial \varrho_2}{\partial \alpha} = -\ln\left(1+\left(1-\alpha\frac{\exp\left(\frac{\varrho_1}{\alpha}\right)-1}{A_1}\right)\frac{A_2}{1-\alpha}\right)+ \\ +(1-\alpha)\frac{1}{1+\left(1-\alpha\frac{\exp\left(\frac{\varrho_1}{\alpha}\right)-1}{A_1}\right)\frac{A_2}{1-\alpha}}\cdot \\ \cdot\left[\left(-\frac{\exp\left(\frac{\varrho_1}{\alpha}\right)-1}{A_1}+\frac{\alpha}{A_1}\exp\left(\frac{\varrho_1}{\alpha}\right)\frac{\varrho_1}{\alpha^2}\right)\frac{A_2}{1-\alpha} \right. \\ \left. -\left(1-\alpha\frac{\exp\left(\frac{\varrho_1}{\alpha}\right)-1}{A_1}\right)\frac{-A_2}{(1-\alpha)^2}\right] \\ =0 \tag{18}$$

This is a transcendental equation, which can only be solved numerically. Figure 3 shows the derivatives for different values of ϱ_1.

As can be seen, there is a unique zero crossing of the derivative in the applicable range for all possible values of the ϱ_1 parameter, and the derivative goes from a positive value to a negative value. In order to get the boundary of the achievable capacity region we have to solve this extreme value problem for any value of the ϱ_1 parameter (the rate for user 1) for the optimal α parameter allocation, and get the value of ϱ_2 for that parameter combination, the maximum achievable rate for user 2 at the given conditions.

VI. THE CAPACITY REGION

Solving (18) numerically for different values for the normalized rate of user1 (ϱ_1), we get the maximum achievable normalized rate for user2 (ϱ_2).

This pair of rates (ϱ_1, ϱ_2 pairs) gives us the boundary of the capacity region on the two-dimensional rate plane. In Figure 4 we plotted this boundary, alongside the school of curves we are getting by fixing the power allocation ratio ($\beta \in [0,1]$) and running the bandwidth allocation ratio ($\alpha \in [0,1]$) the entire range. One can see that these iso-power-ratio curves are touching the boundary only one point each. That point corresponds to the optimal allocation of the other parameter α for the choose value of parameter β.

In the case of differing signal-to-noise ratios, the boundary is clearly not a straight line. That would correspond to the case when we chose the same value for both of the resource allocation parameters: $\alpha = \beta$. This is the optimal allocation only when both users have the same channel conditions and the same receiving signal-to-noise ratio. The more asymmetrical the channel condition for the users, the more curved the boundary will be. This means that when we want to compare the advantage of a given NOMA communication scheme, we have to compare to this curved boundaries of the OMA scheme.

VII. CONCLUSIONS

We have formulated the problem of finding the achievable capacity region, which is the pair of rates for the two users in the case when the base station transmits signals to both

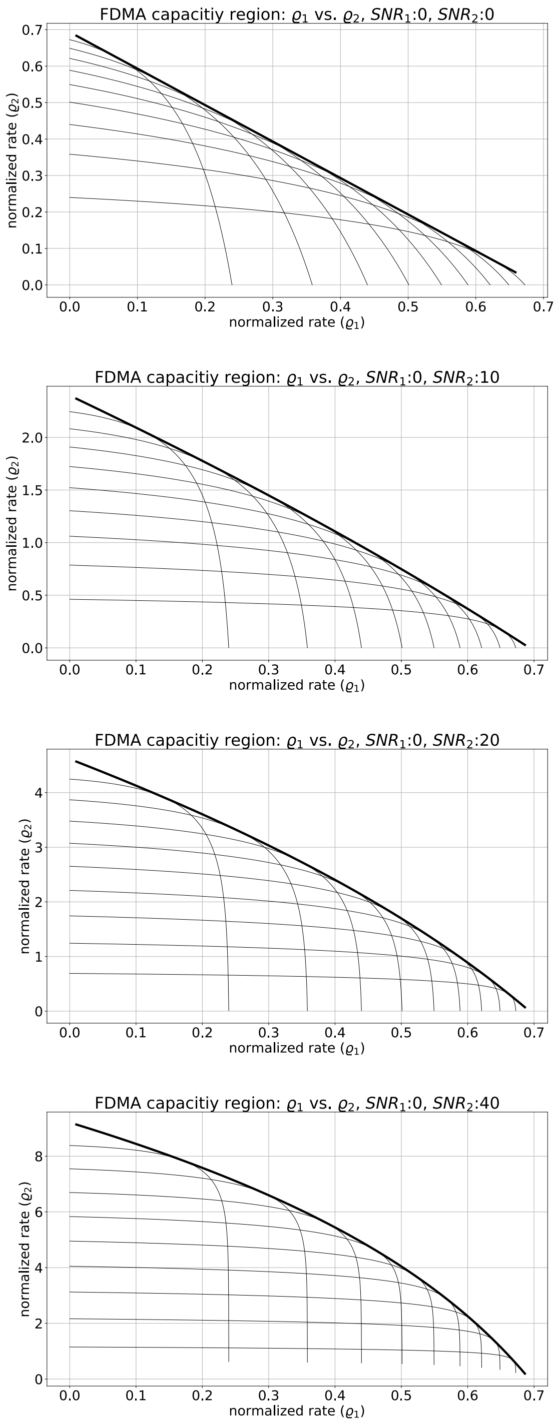

Figure 4. The rate pairs for different bandwidth and power distribution, and the limit of the capacity region

users (downlink communication) and uses frequency division multiple access scheme. In this case, we must share both the available bandwidth and the available transmission power of the base station between the two users in some ratio, denoted by $\alpha \in [0,1]$ and $\beta \in [0,1]$ respectively in this paper. It is far from trivial to determine which pairs of allocation parameters gives us the best result and what the boundary of the achievable capacity region is.

The first approach idea could be to allocate the two parameters at the same ratio ($\alpha = \beta$), which would give us a linear boundary connecting the extreme points (where all the resources are allocated to one or the other user). However, as we have seen, it would not be optimal when the two users' channel conditions differ. We have to solve an extreme value problem, finding the combination of parameters that gives us the maximum value of the rate for one of user while the other user's rate is fixed.

By careful formulation of the rate functions, we managed to eliminate one of the free parameters by fixing one user's rate as a parameter and find the maximum value of the rate of the other user with respect to the other free parameter. The derivation leads to a transcendental equation, which we solved numerically for different values of the first user rate across the applicable range. This allows us to plot the boundary of the capacity region on the rate pairs plane.

References

[1] Y. Liu, Z. Qin, M. Elkashlan, Z. Ding, A. Nallanathan, and L. Hanzo, "Nonorthogonal Multiple Access for 5G and Beyond," *Proceedings of the IEEE*, vol. 105, no. 12, pp. 2347–2381, 2017, DOI: 10.1109/JPROC.2017.2768666.

[2] K. Higuchi and A. Benjebbour, "Non-orthogonal multiple access (noma) with successive interference cancellation for future radio access," *IEICE Transactions on Communications*, vol. 98, no. 3, pp. 403–414, 2015, DOI: 10.1587/transcom.E98.B.403.

[3] L. Dai, B. Wang, Y. Yuan, S. Han, I. Chih-Lin, and Z. Wang, "Non-orthogonal multiple access for 5g: solutions, challenges, opportunities, and future research trends," *IEEE Communications Magazine*, vol. 53, no. 9, pp. 74–81, 2015, DOI: 10.1109/MCOM.2015.7263349.

[4] L. Dai, B. Wang, Z. Ding, Z. Wang, S. Chen, and L. Hanzo, "A survey of non-orthogonal multiple access for 5g," *IEEE communications surveys & tutorials*, vol. 20, no. 3, pp. 2294–2323, 2018, DOI: 10.1109/COMST.2018.2835558.

[5] T. M. Cover and J. A. Thomas, "Network information theory," *Elements of information theory*, vol. 1, no. 1, pp. 509–612, 2005, DOI: 10.1002/047174882X.ch15.

[6] D. Tse and P. Viswanath, *Fundamentals of wireless communication*. Cambridge university press, 2005, DOI: 10.1017/CBO9780511807213.

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