Analysis and Modeling of Very Large Network Topologies

(Invited Paper)

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***Abstract*—We consider the issue of modeling huge, random network topologies that are too large to capture in full details. Such enormous, hard-to-describe network topologies are becoming ubiquitous in numerous settings. The Internet and its logical overlay networks, such as the World Wide Web, as well as online social networks, are well known examples. At the same time, extensive and rapidly growing wireless ad hoc and sensor networks also lead to hard topology modeling questions. In the current paper we primarily focus on large, random wireless networks. We provide a common generalization of various models that covers a number of known models as special cases. We also demonstrate that such a higher level abstraction, despite its very general nature, can still be meaningfully analyzed, and offers quite useful and unique help in solving certain hard networking problems.**

I. INTRODUCTION

Many of the communication networks that we use today, or expect to use in the future, have enormous size. This applies not only to the physical networks, including the Internet as well as emerging ubiquitous wireless networks and large scale sensor networks, but also, or even more, to logical overlay networks, such as the World Wide Web. For example, the number of web pages, according already to a 2006 article [31], was as high as 53.7 billion, already at the time of writing that study. Out of the 53.7 billion, 34.7 billion web pages were indexed by Google. Since then, these numbers grew even further. Beyond the sheer size, the usage of these networks is also expected to be extremely heterogeneous, encompassing a huge number of different applications, traffic patterns, diverse requirements and areas, including business, science, learning, entertainment, social networking and a great many more. At the same time, their physical basis is also heterogeneous, including wired, wireless, optical subnetworks. All this is expected to eventually merge into a ubiquitous, global socio-technical infrastructure.

To understand and reason about huge socio-technical networks, including methods for designing/optimizing them, the traditional network analysis and modeling approaches are generally insufficient, due to their *limited scalability.* Simulation is usually feasible only up to a rather limited network size. Conventional analysis methods, such as teletraffic theory, queuing network modeling etc., also face an uphill battle, quickly losing ground in huge networks. At the same time, modeling and analysis is still indispensable, since one may not be able to experiment with the different variants of a new solution via large scale practical deployment, as it can have a prohibitive cost.

This situation, in which one deals with networks of practically infinite size, has naturally led to the emergence of novel analysis and modeling approaches. They can generally be characterized by having a more abstract, "bird's eye" view of the network and often relying on *asymptotic analysis* on the mathematical side. The special advantage of the asymptotic analysis is that it converts the growing size from a foe to a friend: the larger, the better, from the asymptotic point of view. While it is clear that such methods cannot help much in *local* technical tasks, such as configuring a specific router, they have their important place in the higher layers of the network modeling hierarchy. In the next section we briefly survey how this approach emerged.

II. HISTORY

The first major wave of work in the considered direction was the **experimental statistical analysis** of the Web graph, in which the nodes are web pages and the edges are the hyperlinks. Several research groups in the late 90s independently observed that the node degrees in this graph are distributed according to a *power law* (Kumar et al. [36], Barabási and Albert [5], [6], Broder et al.[12]). Similar phenomena were observed by Faloutsos et al. [19] in the physical Internet topology. All this was in sharp contrast with traditional random graph models that have independent edges, and exhibit (asymptotically) Poisson node degree distributions. The latter models are known as Erdős-Rényi random graphs.

To describe the observed network structure, Barabási and Albert [5] coined the term *"scale-free network"*, based on the observation that in a power law distribution the rescaling of the considered quantity preserves the same power law, changing it only with a constant factor. This quickly became very popular, and triggered the statistical analysis of "scale-freeness" of network topologies not only in (physical or logical) communication networks, but also in networks that arise in biology, genetics, epidemiology, linguistics, electric power distribution, social sciences and in many other areas; see, e.g., the books [10], [11], [13], [14], [42], and hundreds of further references therein.

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In retrospect, one may say that "scale-free networks" generated somewhat more hype than substance. It was rightfully pointed out, e.g., by Li et al. [40] and Alderson et al. [4] that the power law degree distribution alone can easily fall short from adequately modeling the Internet topology, if no other domain specific knowledge is applied.

A parallel major wave of research was to create **generative models**. In contrast to experimental statistical analysis, generative models aim at explaining the observed network structures, and provide algorithmic approaches to generate them, also offering the opportunity for in-depth mathematical analysis. The first such model that became well known was the *Preferential Attachment* model of Barabási and Albert [5]. This model generates a graph such that new nodes are more likely to get connected to those that already have a higher degree. Although the authors did not provide a rigorous analysis, only an approximate reasoning, based on the mean-field approach of physics, the model certainly had intuitive appeal ("the rich get richer" principle). This model had an explanatory power and generated scale-free graphs, so it triggered many follow-up investigations. The first truly rigorous formulation and deep analysis of a preferential attachment model, called Linearized Chord Diagram (LCD) model, was provided by Bollobás at al. [9].

Since then, numerous static and evolving graph models of networks have been proposed and analyzed, both experimentally and with serious mathematical rigor, primarily focusing on asymptotic properties. A few examples are: the ACL model of Aiello, Chung and Lu [2]; the copying model of Kleinberg et al. [33]; the growth-deletion model of Chung and Lu [15]; the self-similar Kronecker-graph model of Leskovec et al.[37]; the compressible Web model of Chierichetti et al. [16]; the forest fire model of Leskovec, Kleinberg and Faloutsos [38]; the geometric preferential attachment model of Flaxman, Frieze and Vera [26]; the spatial preferential attachment model of Aiello et al. [1]; the random perturbation model of Flaxman [25]; as well as a large number of other models and variants, with a lot of intellectual power in their analysis.

About the same time when the above outlined investigations began, another independent wave of asymptotic network modeling was initiated by Gupta and Kumar [29]. This direction focused on analyzing the **scalability of large wireless networks**, primarily ad hoc and sensor networks, from the viewpoint of fundamental limits for transport capacity and related properties. This line of research also attracted much attention. Interestingly, and unfortunately, most of the results are negative. Specifically, they show under various conditions, that the achievable per node throughput tends to 0 with growing network size. Even maintaining global network connectivity is impossible under rather general conditions, if we want to apply nodes with finite processing capacity, see Faragó [21]. Nevertheless, there are also notable exceptions, utilizing various effects, such as mobility (Grossglauser and Tse [28]), restricted traffic pattern (Li et al. [39]), using infrastructure (Liu, Liu and Towsley [41]), or relaxing the condition of full connectivity (Dousse, Franceschetti and Thiran [18], Faragó [24]), to mention only a few examples. Therefore, the issue of wireless network scalability is still under further research.

The graph models that are used in the wireless network investigations are very different from the Internet and Web models. The random graphs in wireless network analysis are based on geometric considerations, and termed **geometric random graphs**. They also have a rich set of analytical results, see, e.g., the books of Franceschetti and Meester [27], and Penrose [44]. In a sense, geometric random graphs are between the classical Erdős-Rényi model and those graph models that are used to describe the Internet and Web topologies. Specifically, geometric random graphs have (asymptotically) Poisson node degree distributions, just like the Erdős-Rényi random graphs. That is, geometric random graphs (modeling wireless network topologies) do *not* exhibit scale-free behavior. On the other hand, their edges are not independent, just like in the Internet/Web models, so they have many properties that are distinctively different from the Erdős-Rényi random graphs.

The current situation. As briefly reviewed above, there exists a vast and rather diverse body of various graph based network models that are mostly analyzed from the viewpoint of asymptotic properties. Note that beyond the theoretical advances they also have emerging practical applications, such as Internet topology generators, search engine optimization, protocol design and optimization in wireless networks etc.

The current situation on the model development and analysis side (which is our primary interest) can be characterized with the following:

- The diversity of models also led to the diversity of analysis methods. With minimal exaggeration one can say that a new analysis method has to be invented almost each time when a new model is proposed. There is a sense of missing unification and a lack of general methods that apply to large families of different models.
- The analysis is often very hard and typically cannot rely on the well developed methods of classical random graph theory, as pointed out by leading experts in the theory of random graphs (Bollobás and Riordan [8]).
- Despite the existence of emerging applications, there is still a large gap between analysis results of descriptive nature and methodology/algorithms that provide meaningful help in network design problems.
- Validation of models is a problem. As pointed out by A.D. Flaxman [25]: "Unfortunately, it is much easier to propose a generative model than to refute one."

III. The Case of Large, Multi-hop Wireless Networks

Wireless networks of large size, random topology and no supporting infrastructure, such as ad hoc and sensor networks, are expected to play an important role in the future. The random network topology of these systems is frequently described by various random graph models, most often by some variant of geometric random graphs. First we review some of the typical classes of graphs that are used in this context.

A. Some Frequently Used Graph Classes for Wireless Network Topology Modeling

An important class is the *Unit Disk Graph (UDG)* [17] model of the network topology. A UDG is a graph that is defined by the (planar) geometry of node positions. It is assumed that each node has the same transmission radius r, and two nodes are connected by a link if and only if they are within distance r (which is often normalized to $r = 1$, hence the name). In other words, the radio range of each node is just a circular disk. As a critical difference from the physical model, in a UDG it does not matter where the rest of the nodes are located and how much interference they generate.

A clear advantage of UDGs is that a number of important algorithmic problems that are NP-complete for general graphs become solvable in polynomial time for this special class [45], thus allowing much more efficient protocols.

Unfortunately, the UDG model is quite simplistic, it is rather far from accurately reflecting the actual radio network topology. A refinement is the *Quasi-Unit Disk Graph (Q-UDG)* model [34], in which a *shrink factor* ρ is added, with $0 < \rho < 1$, for describing the radio range of a node by two concentric circular disks, the outer one with radius r, and the inner one shrunk by the factor ρ, yielding radius ρr. If two nodes are at most ρr distance apart, then they are always connected by a link. If they are more than r apart, then they are never connected. Finally, if the distance is between ρr and r, then the link may or may not exist. Geometrically this means that the radio range of a node can have arbitrary shape, but moderated by the requirement that it should be between a circumscribed circle of radius r and an inscribed circle of radius ρr.

A nice feature of Q-UDGs is that, while providing a more general network topology model, they still preserve the algorithmic advantages of UDGs, at the price of an additional $1/\rho^2$ factor in complexity [34]. Thus, if the shrink factor ρ is a not too small constant, then most of the UDG advantages carry over, with only a constant factor penalty in complexity.

Another natural issue is that different nodes may transmit with different power, or have different spectrum-dependent attenuation of the transmission signal [3]. This leads to the concept of *Disk Graph (DG)*, which differs from the UDG in that each node i has its own, possibly different, transmission radius r_i, and two nodes are connected by an undirected link if they are mutually in each other's range. DGs are somewhat less friendly from the algorithmic point of view than UDGs and Q-UDGs, but still better than general graphs and still allow efficient solutions or approximations for a number of algorithmic problems, as we investigated in [46].

Similarly to the generalization that leads to the Q-UDG concept, one can also introduce *Quasi-Disk Graphs (Q-DG)*, by adding a shrink factor ρ that allows to refine the radio range description as for Q-UDG.

All the above graph models can naturally be extended to higher dimensions, replacing the disks by balls in the appropriate space.

A common nontrivial generalization of all these graphs, the *Bounded Independence Graph (BIG)* model is also worth mentioning [45]. (It is also referred to as *Bounded Growth Graph* [35].) This class is defined by the requirement that the maximum number of independent nodes[1] within the k-hop neighborhood $\mathcal{N}_k(v)$ of any node v is bounded by a polynomial of k. Although this definition is based purely on the graph structure and does not have a direct geometric meaning, it can still be related to geometry through the concept of *doubling metric spaces* [45]. These are metric spaces in which any ball of radius r can be covered by a finite number of balls of radius $r/2$. This property does not hold for all metric spaces, although it holds for Euclidean spaces of any finite dimension[2]. It can be shown that if a geometric graph is defined in a doubling metric space, in analogy with UDG or DG, then it is always a Bounded Independence Graph [45]. A nice feature of this class is that a number of hard algorithmic problems become efficiently solvable in it [35].

So far we have described these classes deterministically, ignoring randomness. Of course, from each graph class one can generate random members, according to various probability distributions. These are usually defined indirectly, through some generating mechanism. For example, if we pick the node positions uniformly at random in a planar domain, e.g., a square, and then construct a UDG over these nodes, then we get a Random Unit Disk Graph.

All these graph classes are related to some kind of geometric insight. It is not surprising, since geometry and distance play a key role in forming the radio network topology. On the other hand, radio propagation (with possible obstacles and other irregularities) can induce much more complicated distances that may not satisfy the mathematical distance axioms, primarily the triangle inequality. Nevertheless, even in this more complicated situation, it is still possible to meaningfully analyze geometric-like graphs and prove nontrivial results about important properties, such as connectivity, as we are going to see in connection with our *Abstract Geometric Random Graphs.*

B. The Issue of Connectivity

Because of the random network topology, it is not at all guaranteed that any two nodes can send messages to each other, as the random graph that represents the network topology may not be connected. To guarantee that all nodes can reach each other, a minimum requirement is that the network topology (which is usually represented by an undirected graph) is *connected.* Since connectivity is a particularly important property, we select it as the focus of our discussion.

Unfortunately, the connectivity requirement is not as innocent as it may look, due to random node positions and limited wireless transmission ranges. It turns out (see, e.g., Gupta and Kumar [29], [30]) that in typical cases, such as placing

[1] A set of nodes in a graph is called independent if there is no edge between any two of them.

[2] Radio propagation properties may lead to a "radio-distance" that is quite different from Euclidean.

the nodes in a planar disk independently and uniformly at random, the price of connectivity is very high: the transmission range needs to be set such that it asymptotically results in an infinitely growing number of neighbors.

This phenomenon is a serious *threat to scalability* in these networks. After all, one cannot expect that a small wireless node with limited power and modest capabilities can serve an unbounded number of neighbors.

One might hope at this point that for different modeling assumptions the situation may perhaps improve. For example, one may try different deployment domains, different probability distributions, different distance metrics, etc. Unfortunately, however, it has been proven in a very general model that none of these can relieve the scalability bottleneck, see Faragó [23]. It appears that unbounded node degrees are unavoidable whenever full connectivity is required in the limit in a random, geometrically induced topology. This is, of course, bad news for hoping a scalable implementation.

It is therefore of keen importance whether better scalability can be achieved if we are willing to give up full connectivity and substitute it with the milder requirement of *partial connectivity*. This means, as a price for keeping the node degrees bounded, we accept that only most, but not all, nodes are in a connected component. The motivation is that in many potential applications, such as a network of randomly placed sensors, it is acceptable to have only a majority (say, 99%) of nodes in a connected component and the rest are possibly disconnected.

We review some results on the fundamental limits related to such partial connectivity, under the most general modeling assumptions we can set up. Specifically, based on our earlier work [22], we exhibit the asymptotically optimal trade-off between the fraction of nodes that can be kept in a connected component as a function of the bound on the expected node degrees.

IV. A Motivating Example

Let us consider a large sensor network. Due to the limited processing capabilities of the small sensor nodes, each one is capable of maintaining connections only to at most three other nodes in our example. The existence of wireless links depends on distance, but the actual form of the dependence is unknown. Moreover, random obstacles to radio waves are also present, and two nodes can only communicate if no such obstacle separates them.

The sensor nodes are distributed in space independently, according to a common, but unknown probability distribution. The locations of the random obstacles are also independent of each other and of the node locations, but otherwise the position, size and shape of each obstacle can have an arbitrary probability distribution, which is again unknown. We only assume that the events that links are blocked by an obstacle can be considered independent.

Without further information about this sensor network, is it possible to provide a nontrivial lower bound on the number of sensors that will be necessarily pushed to the "periphery"? By periphery we mean those nodes that are not part of the largest connected component of the network topology.

The traditional approach to answer this question would be to specify the probability distributions and other parts of the model (such as how link existence depends on distance, etc.), and then do (tedious) calculations under the specific conditions. If, however, anything changes in the conditions, the results may not carry over. Our general approach will make it possible to avoid this, and provide a nontrivial bound that is valid for all practically relevant cases.

V. Random Graph Models in the Most General Setting

In order to build up our modeling approach, let us first explain what we mean by random graphs and a random graph model in the possibly most general sense.

In full generality, by a *random graph* on a fixed number of nodes (n) we mean a random variable that takes its values in the set of all undirected graphs on n nodes. We are going to denote by G_n a random graph on n nodes. At this point, it is still completely general, possibly generated by any mechanism, with arbitrary dependencies among its parts, it is just *any* graph-valued random variable, taking its values among undirected graphs on n nodes.

Having introduced general random graphs, a *random graph model* is given by a sequence of graph valued random variables, one for each possible value of n:

$$\mathcal{M} = (G_n;\ n \in \mathbf{N}).$$

Next we introduce some general features that apply to any random graph model.

A. Degrees and Connectivity

Let G_n be any random graph on n nodes and let us denote by $e(G_n)$ the number of edges in the graph. We characterize the degrees of G_n by the expected degree of a randomly chosen vertex, which we call he *expected average degree* of G_n. It is denoted by $\overline{d}(n)$ and defined by

$$\overline{d}(n) = \frac{2\mathrm{E}(e(G_n))}{n}.$$

It is based on the fact that the actual average degree in any graph G on n nodes is $2e(G)/n$. Often the expected degree of each individual node is also equal to $\overline{d}(n)$, but in a general model it may not hold. (Note that even if the expected degree of each node is equal to the expected average degree, it does not mean that the *actual* random degrees are also equal.)

Ideally, we would like a random graph model in which $\overline{d}(n)$ remains bounded by a constant and the model is *asymptotically almost surely (a.a.s.)* connected, meaning

$$\lim_{n\to\infty} \Pr(G_n \text{ is connected}) = 1.$$

Note: Whenever we write down a limit, such as the one above, we also assume that the limit exists.

Since, as mentioned in Section III-B, asymptotic connectivity is not possible in most models without unbounded degrees, therefore, one may hope that if less than full connectivity is required, then there is a better chance to keep the node degrees bounded. To this end, let us define a weaker version of connectivity.

Definition 1: **(β-connectivity)** *For a real number $0 \leq \beta \leq 1$, a graph G on n nodes is called β-connected if G contains a connected component on at least βn nodes.*

When we consider a sequence of graphs with different values of n, then the parameter β may depend on n. When this is the case, we write β_n-connectivity. Note that even if $\beta_n \to 1$, this is still weaker than full connectivity in the limit. For example, if $\beta_n = 1 - 1/\sqrt{n}$, then we have $\beta_n \to 1$, but each graph on n nodes can still have $n - \beta_n n = \sqrt{n}$ nodes that are not part of the largest connected component.

VI. Abstract Geometric Random Graph Models

Let us now introduce a model class that reflects a typical feature of geometric random graph models. This feature is that in geometric random graphs the primary random choice is picking random nodes from some domain and then the edges are already determined by some geometric property (typically some kind of distance) of the random nodes. We elevate this approach to an abstract level that includes many special cases of interest. The most general version of our abstract geometric model is built using the components detailed below.

A. Representing the Nodes: Node Variables

The nodes are represented by an infinite sequence $X_1, X_2, \ldots$ of random variables, called *node variables*. They take their values in an arbitrary (nonempty) set S, which is called the *domain* of the model. In most practical cases the domain is a simple subset of the Euclidean plane or of the 3-dimensional space. In general, however, S can be any abstract set from which we can choose random elements[3]. When we want to generate a random graph on n nodes, then we use the first n entries of the sequence, that is, $X_1, \ldots, X_n$ represent the nodes in G_n. It is important to note that we do not require the node variables to be independent.

B. Representing the Links: Edge Functions

We denote by $Y_{ij}^{(n)} \in \{0, 1\}$ the indicator of the edge between nodes X_i, X_j in the random graph G_n. Since loops are not allowed, we always assume $i \neq j$, without repeating this condition each time. The (abstract) geometric nature of the model is expressed by the requirement that the random variables $Y_{ij}^{(n)}$ are determined by the nodes $X_1, \ldots, X_n$, possibly with additional independent randomization. Specifically, we assume that there exist functions $f_{ij}^{(n)}$, such that

$$Y_{ij}^{(n)} = f_{ij}^{(n)}(X_1, \ldots, X_n, \xi_{ij})$$

where ξ_{ij} is a random variable that is uniformly distributed on $[0, 1]$ and is independent of all the other defining random variables of the model (i.e., the node variables and all the other ξ_{kl} variables). Henceforth the role of ξ_{ij} is referred to as *independent randomization*[4]. The undirected nature of the graph is expressed by the requirement $Y_{ij}^{(n)} = Y_{ji}^{(n)}$, which can simply be enforced by computing all values for $i < j$ only and defining the $i > j$ case by exchanging i and j.

C. Restrictions

Regarding the abstract geometric random graph model in the presented very general form, it is clear that allowing *totally arbitrary* node variables and edge functions offers little hope for meaningful analysis. Therefore, next we introduce some restricting conditions. Later we are going to see that one has to make only surprisingly mild restrictions to meaningfully analyze the trade-off between node degrees and β-connectivity.

1) Locality: Up to now we allowed that an edge in G_n can depend on all the nodes, and the dependence expressed by the $f_{ij}^{(n)}$ functions can be arbitrary and different for each edge. To get a little closer to the usual geometric random graph model, let us introduce the following property, called *locality*. Informally, it restricts the dependence of an edge to its endpoints, in a homogeneous way, but still via an *arbitrary* function.

Definition 2: **(Locality)** *An abstract geometric random graph model is called* local*, if for every n and $i, j \leq n$ the existence of an edge between X_i, X_j depends only on these nodes. Moreover, the dependence is the same for every i, j, possibly with independent randomization. That is, there are functions $f^{(n)}$ such that the edge indicators are expressible as*

$$Y_{ij}^{(n)} = f^{(n)}(X_i, X_j, \xi_{ij})$$

where ξ_{ij} represents the independent randomization.

2) Name Invariance: Our second condition called *name invariance* refers to the joint distribution of nodes. If we allow totally arbitrary joint distribution, then it offers little chance for meaningful analysis. On the other hand, restricting ourselves only to independent, identically distributed (i.i.d.) node variables would exclude important cases, such as clustering. Therefore, we introduce a condition that allows more general than i.i.d. node variables, but still makes meaningful analysis possible. To introduce it, let us first recall a useful concept from probability theory, called exchangeability.

Definition 3: **(Exchangeable random variables)** *A sequence of random variables is called* exchangeable *if for any $k \geq 1$, it holds that if we select any k of the random variables, the joint distribution of the selected random variables depends only on k, but is independent of which particular k variables are selected, and in which order.*

Note that i.i.d. random variables are always exchangeable, but the converse generally does not hold, so exchangeable random variables form a larger family.

[3]To avoid mathematical complications that would only obscure the main message, we assume that all considered sets, functions etc. are measurable with respect to the used probability measures and all considered expected values exist. This is satisfied in in every practically relevant model.

[4]Note that the specified distribution of ξ_{ij} does not impose a restriction, since the functions $f_{ij}^{(n)}$ are arbitrary.

Now let us introduce the condition that we use to restrict the arbitrary dependence of node variables.

Definition 4: **(Name invariance)** *An abstract geometric random graph model is called* name invariant*, if its node variables are exchangeable.*

We call it the *name invariance* of the model because it means the names (the indices) of the nodes are irrelevant in the sense that the joint probabilistic behavior of any fixed number of nodes is invariant to renaming (reindexing) the nodes. In particular, it also implies that the individual node variables are identically distributed, but they do not have to be independent.

Name invariance is naturally satisfied with the most frequently used random node choices, such as uniform independent random points in a planar domain, or a Poisson point process in the plane, or in higher dimension. We allow, however, much more complex node generation (over an arbitrary set!) since dependencies are not excluded by name invariance.

A simple example for a dependent, yet still name invariant, node generation process is a "clustered uniform" node generation. As an example, let S be a sphere in 3-dimensional space, i.e., the surface of a 3-dimensional ball. Let R be the radius of the ball. Let us first generate a pivot point Y uniformly at random from S. Then generate the nodes $X_1, X_2, \ldots$ uniformly at random and independently of each other from the neighborhood of radius $r \ll R$ of the random pivot point Y (on the sphere). It is directly implied by the construction that exchangeability holds. Moreover, any particular X_i will be uniformly distributed over the *entire* sphere, since Y is uniform over the sphere. On the other hand, the X_i are far from independent of each other, since they cluster around Y, forcing any two of them to be within distance $2r$. The setting can be generalized to applying several pivot points and non-uniform distributions, creating a more sophisticated clustering.

VII. Specific Classes within Abstract Geometric Random Graphs

Before turning to results, let us present some examples to show the usefulness and comprehensiveness of the generalization provided by our abstract geometric random graphs. These examples illustrate that most practically relevant models for wireless network topologies fit in the common generalization that we provided by introducing abstract geometric random graphs.

A. Geometric Random Graphs

All the usual geometric random graph models fit naturally in our general framework. For example, the base set S can be chosen as a unit disk or square in the plane or a unit ball or cube (or any other domain) in higher dimension. Let us choose i.i.d. points $X_1, X_2, \ldots$ from S, according to some probability distribution. Let $\rho(x, y)$ denote the distance of the points $x, y \in S$, it can be any distance function. Finally, let $r > 0$ be a radius (possibly depending on n). Then the edge function

$$f^{(n)}(X_i, X_j, \xi_{ij}) = \begin{cases} 1 & \text{if } \rho(X_i, X_j) \le r \\ 0 & \text{if } \rho(X_i, X_j) > r \end{cases} \qquad (1)$$

defines a geometric random graph in the usual sense. (The independent randomization is not used here, so the edge function does not depend on ξ_{ij}.) It is clear that this includes all the usual geometric random graph models, allowing any metric space as the basis. Moreover, we can also use non-independent points, such as the "clustered uniform" example in the previous section, as long as the distribution is exchangeable.

B. Erdős-Rényi Random Graphs

The by now classical random graph model of Erdős and Rényi (see, e.g., [7], [32]), where each possible edge is included independently with some probability p is also included as a direct special case. We can set $S = \{1, \ldots, n\}$ and for $X_i, X_j \in S$

$$f^{(n)}(X_i, X_j, \xi_{ij}) = \begin{cases} 1 & \text{if } \xi_{ij} \le p \\ 0 & \text{if } \xi_{ij} > p \end{cases}$$

Note that now the edge function depends only on the independent randomization, so indeed each edge is included independently with probability p.

C. Geometric But Non-Metric Example: Battery Levels

In the geometric random graph models ρ satisfies the triangle inequality. This, however, cannot capture all situations that occur in ad hoc or sensor networks. As an example, assume the nodes are located in the plane. Let x_i, y_i be the coordinates of the i^{th} node. Furthermore, we also characterize a node with its battery level $E_i > 0$. E_i represents the remaining energy, assuming the node is not fully out of energy. Thus, a node is represented by a triple $X_i = (x_i, y_i, E_i)$. Let $d(E_i)$ be the distance over which a node can communicate, given its energy level E_i. (The function $d(E_i)$ can be derived from the physical characteristics of the node and from radio propagation conditions.) Now, a possible example of such a "distance" function is

$$\rho_1(X_i, X_j) = \frac{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}{\min\{d(E_i), d(E_j)\}}$$

If we take $r = 1$ and use the above ρ_1 function in (1), then it expresses the condition that a link exists if and only if its end nodes are at most at a distance that can be bridged by the energy levels of both nodes. Note that the above function ρ does not satisfy the triangle inequality, so it does not lead to a geometric random graph model in the usual sense. On the other hand, it still fits in our framework, as in (1) we did not require the triangle inequality to hold for ρ.

D. Another Non-Metric Example: Link Blocking

We can capture some features of traffic dependent network characteristics, as well. Let each node i be characterized by a triple $X_i = (x_i, y_i, \lambda_i)$, where x_i, y_i are planar coordinates and λ_i is the traffic demand of the node. Let B_{ij} be the blocking probability of the link (i, j), given that the link exists. We may compute B_{ij} as a function of λ_i, λ_j from some traffic model. For example, if we use Erlang's well known formula,

assuming a capacity of C units on the link and its load is taken as the sum of its end nodes' traffic load $\lambda_i + \lambda_j$, then we obtain

$$B_{ij} = \frac{(\lambda_i + \lambda_j)^C / C!}{\sum_{i=0}^{C} (\lambda_i + \lambda_j)^i / i!}.$$

(Of course, we may use other traffic models, as well, this is just an example.) Now we can take the "distance" function

$$\rho_2(X_i, X_j) = \frac{1}{1 - B_{ij}} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

and use it in (1) with some radius r. We can observe that for small blocking probability ($B_{ij} \ll 1$) $\rho_2(X_i, X_j)$ will be approximately the same as the Euclidean distance. On the other hand, as B_{ij} approaches 1, the factor $\frac{1}{1-B_{ij}}$ tends to infinity and, therefore, high blocking probability makes the existence of the link in the model less likely, even if the physical distance is small. This example also violates the triangle inequality, so it is not a geometric random graph.

E. Log-Normal Shadowing

A typical phenomenon in the radio environment is *fading*. An example of fading is a relatively slow random fluctuation in the signal strength, which occurs even if the locations are fixed. Measurements show that this random variation can be accurately modeled by a log-normal distribution (see, e.g., [43]). Hence the name *log-normal shadowing*, which is widely used for this phenomenon. A way to capture it in our model is this. Let us characterize a node i by a triple $X_i = (x_i, y_i, \eta_i)$, where x_i, y_i represent a random position in the plane and each η_i is an infinite sequence of independent, log-normally distributed random variables:

$$\eta_i = (\eta_j^{(i)};\ j = i, i+1, i+2, \ldots).$$

The "distance" is defined as

$$\rho_3(X_i, X_j) = \eta_b^{(a)} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

where $a = \min\{i, j\}$ and $b = \max\{i, j\}$. (The reason for we need an infinite sequence of log-normal random variables is that this way we can have independent log-normal shadowing for every link.) This distance can express the fact that from the radio communication point of view we really perceive an "effective distance", which is a log-normally modulated random variant of the physical distance. Using this ρ_3 in (1) leads again to a random graph that is not geometric, as ρ does not satisfy the distance axioms.

F. Directional Antennas

We can also represent directional antennas in the model. As a simple example, let Y_i be the position of a node in the Euclidean plane, α_i be the angle (with respect to some fixed coordinate axis) at which its antenna is directed, and δ_i be the angular width of the beam (assuming an idealized directional antenna). Let us represent the node by the variable $X_i = (Y_i, \alpha_i, \delta_i)$. Let $S(X, \alpha, \delta)$ denote the planar angular sector pointed at X, with its axis of symmetry directed at α and of angular width δ. Further, let $||.||$ denote the Euclidean norm. Then we can introduce the following "distance":

$$\rho_4(X_i, X_j) = \begin{cases} ||Y_i - Y_j|| & \text{if } X_i \in S(X_j, \alpha_j, \delta_j) \text{ and} \\ & \quad X_j \in S(X_i, \alpha_i, \delta_i) \\ \infty & \text{otherwise} \end{cases}$$

If we use this function $\rho_4(X_i, X_j)$ in (1), then we get a model of a random ad hoc network topology with directional antennas.

G. Terrain Variations, Obstacles

Another example is to take into account uneven radio propagation characteristics due to terrain variations or propagation obstacles. For example, let us assume that the nodes operate in a frequency band in which only line of sight communication is possible (such as infrared). Then two nodes can only communicate if there is no obstacle covering them from each other. This feature can also be built into the model. Let X_i be the plane position of a node. Assume there exists a set $\mathcal{R} = \{R_1, R_2, \ldots\}$ of random obstacles in the area. Let $s(X_i, X_j)$ be the line segment connecting the points X_i, X_j, and let $L(X_i, X_j, \mathcal{R})$ be the "line of sight" function:

$$L(X_i, X_j, \mathcal{R}) = \begin{cases} 1 & \text{if } s(X_i, X_j) \cap R_k = \emptyset \ \ (\forall k) \\ \infty & \text{otherwise} \end{cases}$$

To express that only those nodes can communicate that are in line of sight of each other, let us introduce the "distance"

$$\rho_5(X_i, X_j) = ||X_i - X_j|| L(X_i, X_j, \mathcal{R}).$$

If this is used in (1), then we get a network topology model that can deal with radio propagation obstacles.

H. Combinations

The various conditions in the preceding examples can be combined into more complex models. For example, if we want that all the conditions expressed by the $\rho_1, \ldots, \rho_5$ functions are satisfied, then we can use

$$\rho(X_i, X_j) = \max\{\rho_1(X_i, X_j), \ldots, \rho_5(X_i, X_j)\}$$

in (1).

VIII. Threshold Function for Partial Connectivity

We define a concept that will characterize the trade-off between node degrees and the type of partial connectivity that we introduced as β-connectivity in Definition 1. For notational convenience, the set of nonnegative real numbers, extended with ∞, will be denoted by $\mathbf{R}_0^\infty$. Real functions are also extended to ∞ by $f(\infty) = \lim_{x \to \infty} f(x)$, whenever the limit exists (it will always exist in our cases). The value of β is always assumed to be in $[0, 1]$.

Let us first explain the threshold function concept informally. We define a threshold for β-connectivity, such that whenever β is above the threshold, then it is impossible to achieve a.a.s. β-connectivity for any model in the considered

family of random graph models. On the other hand, if β is below the threshold, then this is not the case anymore, that is, there is at least one model in the family that achieves a.a.s β-connectivity with this β. Thus, the threshold separates the cases when a.a.s. β-connectivity is impossible, from the cases when it is possible. Since the threshold will depend on the expected average degree, we call it threshold function.

Now let us present the formal definition. Recall that the expected average degree in a random graph G_n is defined as $\overline{d}(n) = 2\mathrm{E}(e(G_n))/n$.

Definition 5: **(Threshold for β-connectivity)** *Let $\mathcal{F}$ be a family of random graph models. For any model $\mathcal{M} \in \mathcal{F}$ let G_n denote the random graph on n nodes generated by $\mathcal{M}$ and let*

$$D_{\mathcal{M}} = \limsup_{n\to\infty} \overline{d}(n)$$

be the limiting expected average degree. A function $f : \mathbf{R}_0^\infty \mapsto [0,1]$ is called a β-connectivity threshold function for $\mathcal{F}$ if the following two conditions are satisfied:

(i) *For any model $\mathcal{M} \in \mathcal{F}$ and for every $\beta > f(D_{\mathcal{M}})$*

$$\lim_{n\to\infty} \Pr(G_n \textit{ is } \beta\textit{-connected}) < 1$$

holds, where G_n is generated by $\mathcal{M}$.

(ii) *If β is below the threshold, then (i) does not hold anymore, in the following sense. For every $\epsilon > 0$ there exists a model $\mathcal{M}_0 \in \mathcal{F}$ and a*

$$\beta \le f(D_{\mathcal{M}_0}) - \epsilon$$

such that

$$\lim_{n\to\infty} \Pr(G_n \textit{ is } \beta\textit{-connected}) = 1$$

where G_n is generated from $\mathcal{M}_0$.

The importance of this concept is the following. If for a considered class $\mathcal{F}$ of random graph models we can find out what the corresponding β-connectivity threshold function is, then we can tell precisely what range of expected average degrees allow a.a.s. β-connectivity for a given β. Or, conversely, if we know the (asymptotic) expected average degree for a particular model $\mathcal{M}$ in the considered class, then we can decide what level of connectivity can be asymptotically achieved for this model.

IX. Computing the Threshold

Now we state the theorem that conveys the surprising message that for the very general class of abstract geometric random graph models we can still find the *precise* β-connectivity threshold function, if we assume that the models satisfy the conditions of locality and name invariance. The previously presented examples all satisfy these conditions, so they show that even with these restrictions we can still include many complex and practically important models. For the proof of the theorem, see [22].

Theorem 1: **(Threshold function for local and name invariant abstract geometric graphs)** *Let $\mathcal{F}$ be the family of local and name invariant abstract geometric random graph models. For any model $\mathcal{M} \in \mathcal{F}$ set*

$$D_{\mathcal{M}} = \limsup_{n\to\infty} \overline{d}(n).$$

Then the β-connectivity threshold function for $\mathcal{F}$ is

$$f(D_{\mathcal{M}}) = 1 - \mathrm{e}^{-D_{\mathcal{M}}}.$$

X. Consequences for Full Connectivity

It is worth mentioning that the definition of the threshold function and Theorem 1 directly imply that bounded expected average degrees in $\mathcal{F}$ exclude a.a.s. β_n-connectivity when $\beta_n \to 1$. As a result, a.a.s. full connectivity, which corresponds to $\beta = 1$, is also excluded. These claims are formally stated below, the proof is a direct application of Theorem 1.

Theorem 2: Let $\beta_n \to 1$ be an arbitrary sequence in $[0,1]$. Then for any local and name invariant abstract geometric random graph model $\mathcal{M}$ it holds that if $D_{\mathcal{M}} < \infty$, then the random graphs generated by $\mathcal{M}$ cannot be a.a.s. β_n-connected.

The interpretation of this result is that (asymptotically) the requirements of full connectivity and bounded degrees are incompatible, in the broad class of models we have considered.

At this point one may wonder whether there is *any* meaningful random graph model at all, in which a.a.s. full connectivity is possible, yet the node degrees remain bounded. Note that our results do not exclude this, since they only apply to local and name invariant abstract geometric random graph models. Although this class is quite comprehensive, it does not contain *all* meaningful models.

A nontrivial example worth mentioning here is the (random) Euclidean minimum spanning tree (MST). Let us choose n i.i.d. random points in the d-dimensional unit cube and view them as vertices of a complete graph, where each edge is assigned a weight that is equal to the (random) distance of its endpoints. Let T_n be the MST of this graph. Note that T_n is unique with probability 1. It is clear that T_n is connected, as, by definition, it is a spanning tree. Moreover, the following nontrivial fact is known: for every fixed dimension the maximum degree of the Euclidean MST is bounded by a constant, depending only on the dimension, but not on n (see, e.g., [47]). Thus, the model $\mathcal{M} = (T_n;\ n \in \mathbf{N})$ has the property that it is fully connected, yet its node degrees remain bounded.

It is clear that the Euclidean MST model is name invariant, since nothing depends on how the nodes are indexed. Does it then contradict to our results? No, because it does not satisfy the requirement of locality. Of course, the usual definition of the MST is indeed not local. But now our results imply that the non-locality is *unavoidable* in this case, as long as we want to preserve name invariance. In other words, it is impossible to define the Euclidean MST in a local way, such that, at the same time, the model is also name invariant.

Note that the fact that the Euclidean MST cannot be defined locally, with name invariance, is nontrivial. For example, one might try to define new node variables that

contain enough information to decide for any pair whether an MST edge connects them, without looking at other nodes. A possibility is to introduce new node variables $Y_i = (X_i, \ldots, X_n, X_1, \ldots, X_{i-1})$, with edges that connect two such new nodes if their first components are connected by an MST edge, among the original X_i variables. In this way we can create a locally defined MST model, since one can decide from Y_i, Y_j alone, whether X_i, X_j are connected by an MST edge, as the information about *all* the original nodes are available in each of the new node variables. Thus, in the transformed domain we have a *local* model. The MST over the Y_i variables will be isomorphic (with probability 1) to the MST over the X_i node variables, so the new model is equivalent to the original, yet it is local. Then, however, the name invariance would be destroyed. Even though each Y_i individually has the same distribution (since it does not matter in what order the X_i are listed), but the joint distribution of Y_1 and Y_2 will not be the same as the joint distribution of Y_1 and Y_3. The reason is that the first coordinate of $Y_1 = (X_1, \ldots, X_n)$ is the same as the last coordinate of $Y_2 = (X_2, \ldots, X_n, X_1)$, but such a relationship does not hold between Y_1 and $Y_3 = (X_3, \ldots, X_n, X_1, X_2)$.

Generally, it follows from our results and from the aforementioned properties of the Euclidean MST that no matter how tricky local definition we invent for this random graph model, it cannot preserve name invariance. The fact that name invariance *excludes* the possibility of a local Euclidean MST definition appears to be hard to prove without our results.

XI. Solving the Motivating Example

In the motivating sensor network example, we observe that the model is described by a local and name invariant abstract geometric graph model, no matter what the unknown probability distributions are. The reason for locality is that once we choose the positions of two sensors, the probability that a link exists between them does not depend on the locations of other sensors. Although it does depend on the obstacles, but they are independent of the sensor positions, and block the links independently. Name invariance also holds in this example, as the sensor positions are i.i.d., which is a special case of an exchangeable system of random variables. The node degree bound of 3 yields $D_{\mathcal{M}} \leq 3$. By Theorem 1, the threshold function for β-connectivity in our case satisfies

$$f(D_{\mathcal{M}}) = 1 - \mathrm{e}^{-D_{\mathcal{M}}} \leq 1 - \mathrm{e}^{-3} \approx 0.95.$$

Thus, we can conclude that despite the very vague information about the system, we are still able to calculate that at least about 5% of the nodes cannot belong to the largest connected component.

Thus, our general result was able to easily come to a conclusion that would otherwise be rather hard to obtain without having further information.

XII. Conclusion

After briefly reviewing a number of models that are used to capture large network topologies, we focused on analyzing a notorious obstacle to wireless network scalability. This obstacle is the phenomenon that in geometrically generated random network topologies the requirement of asymptotic full connectivity results in infinitely growing expected node degrees. To address the issue, we have set up a general modeling framework, the abstract geometric random graph model. This contains many different possible random graph models as special cases. In this framework we can quantify the precise trade-off between the expected node degrees and the fraction of nodes that can belong to the largest connected component.

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